Maxwell's Equations

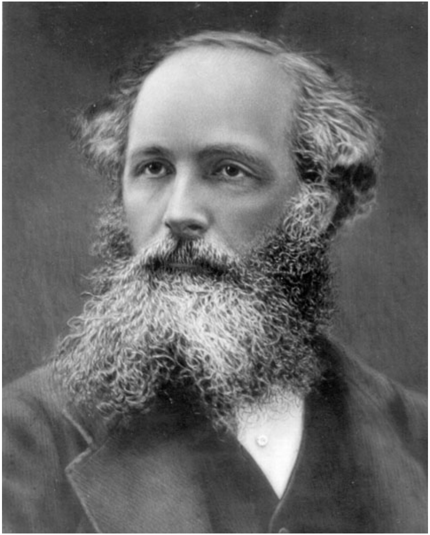
Peter Hertel

University of Osnabrück, Germany

Lecture presented at APS, Nankai University, China

http://www.home.uni-osnabrueck.de/phertel

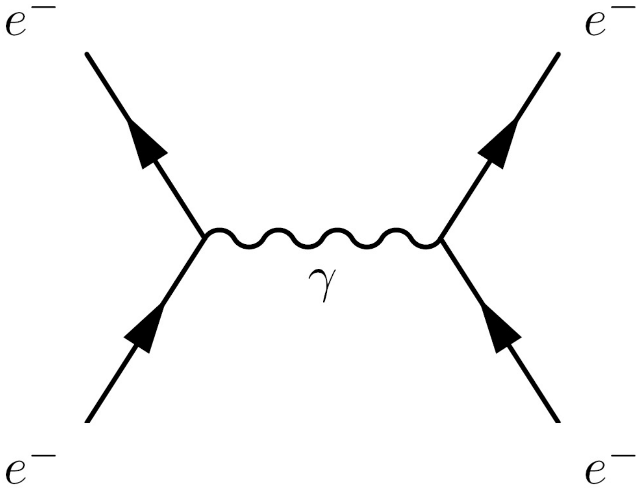
Spring 2012



James Clerk Maxwell, 1831-1873

The electromagnetic field

- the electromagnetic field is felt by charged particles
- charged particles are accelerated
- the Lorentz force
  $\dot{\boldsymbol{p}} = q\{\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\}$ where $\boldsymbol{p} = \dfrac{m\boldsymbol{v}}{\sqrt{1-(v/c)^2}}$
- $\boldsymbol{E} = \boldsymbol{E}(t, \boldsymbol{x})$ is the electric field
- $\boldsymbol{B} = \boldsymbol{B}(t, \boldsymbol{x})$ is the induction field
- the Lorentz force defines the electromagnetic field



Electron-electron scattering

The em-field is produced by charged particles

- close-by charges will have close-by effect
- charge density $\varrho = \varrho(t, \boldsymbol{x})$
- current density $\boldsymbol{j} = \boldsymbol{j}(t, \boldsymbol{x})$
- charge conservation:
  $$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \mathrm{d}V\,\varrho + \int_{\partial V} \mathrm{d}\boldsymbol{A}\cdot\boldsymbol{j} = 0$$
- continuity equation $\nabla_t \varrho + \boldsymbol{\nabla}\cdot\boldsymbol{j} = 0$
- ϱ and $\boldsymbol{j}$ produce the electromagnetic field $\boldsymbol{E}, \boldsymbol{B}$
- but how?

De nihilo nihil

From nothing (follows, comes) nothing.

- as simple aus possible
- differential equations
- equations should be valid in all inertial systems
- respect space inversion symmetry $P : \boldsymbol{x} \to -\boldsymbol{x}$
- respect time inversion symmetry $T : t \to -t$
- try it with fields and first order derivatives
- try it with scalars and vectors only
- superposition principle
- the electromagnetic field may exist in vacuum
- electric charge is conserved

Transformation properties

S/V	T	P	field	∇_t	grad	div	curl
s	+	+	ϱ			$\boldsymbol{\nabla}\cdot\boldsymbol{E}$	
s	+	-					
s	-	+		$\nabla_t\,\varrho$		$\boldsymbol{\nabla}\cdot\boldsymbol{j}$	
s	-	-				$\boldsymbol{\nabla}\cdot\boldsymbol{B}$	
v	+	+		$\nabla_t\,\boldsymbol{B}$			$\boldsymbol{\nabla}\times\boldsymbol{E}$
v	+	-	$\boldsymbol{E}$	$\nabla_t\,\boldsymbol{j}$	$\boldsymbol{\nabla}\varrho$		
v	-	+	$\boldsymbol{B}$				$\boldsymbol{\nabla}\times\boldsymbol{j}$
v	-	-	$\boldsymbol{j}$	$\nabla_t\,\boldsymbol{E}$			$\boldsymbol{\nabla}\times\boldsymbol{B}$

- linear relations betwenn entries in the same row

Almost Maxwell's equations

S/V	T	P	field	∇_t	grad	div	curl
s	+	+	ϱ			$\boldsymbol{\nabla} \cdot \boldsymbol{E}$	
s	+	-					
s	-	+		$\nabla_t\, \varrho$		$\boldsymbol{\nabla} \cdot \boldsymbol{j}$	
s	-	-				$\boldsymbol{\nabla} \cdot \boldsymbol{B}$	
v	+	+		$\nabla_t\, \boldsymbol{B}$			$\boldsymbol{\nabla} \times \boldsymbol{E}$
v	+	-	$\boldsymbol{E}$	$\nabla_t\, \boldsymbol{j}$	$\boldsymbol{\nabla}\varrho$		
v	-	+	$\boldsymbol{B}$				$\boldsymbol{\nabla} \times \boldsymbol{j}$
v	-	-	$\boldsymbol{j}$	$\nabla_t\, \boldsymbol{E}$			$\boldsymbol{\nabla} \times \boldsymbol{B}$

- $\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \varrho$
- $\nabla_t\, \varrho + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$
- $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$
- $\boldsymbol{\nabla} \times \boldsymbol{E} + \alpha\, \nabla_t\, \boldsymbol{B} = 0$
- $(1/\mu_0) \boldsymbol{\nabla} \times \boldsymbol{B} - \beta\, \epsilon_0\, \nabla_t\, \boldsymbol{E} = \boldsymbol{j}$

Maxwell's equations

1. $\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \varrho$ ✓
2. $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ ✓
3. $\boldsymbol{\nabla} \times \boldsymbol{E} + \alpha \nabla_t \boldsymbol{B} = 0$: with $\alpha = 1$ ✓
4. $(1/\mu_0)\boldsymbol{\nabla} \times \boldsymbol{B} - \beta \epsilon_0 \nabla_t \boldsymbol{E} = \boldsymbol{j}$: with $\beta = 1$ ✓

- divergence of (4) with (1) results in $\beta = 1$ since $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{A}) = 0$
- to show $\alpha = 1$ is more difficult

Relatively moving inertial frames

- $t' = t$ and $\boldsymbol{x}' = \boldsymbol{x} + \boldsymbol{u}t$
- for $|\boldsymbol{u}| \ll c$ no difference between Galilei and Lorentz transformation
- $\boldsymbol{E}'(t', \boldsymbol{x}') + \boldsymbol{v}' \times \boldsymbol{B}'(t', \boldsymbol{x}') = \boldsymbol{E}(t, \boldsymbol{x}) + \boldsymbol{v} \times \boldsymbol{B}(t, \boldsymbol{x})$
- $\boldsymbol{v}' = \boldsymbol{v} + \boldsymbol{u}$
- $\boldsymbol{B}'(t, \boldsymbol{x}') = \boldsymbol{B}(t, \boldsymbol{x})$ and $\boldsymbol{E}'(t', \boldsymbol{x}') = \boldsymbol{E}(t, \boldsymbol{x}) - \boldsymbol{u} \times \boldsymbol{B}(t, \boldsymbol{x})$
- $\boldsymbol{\nabla}' \times \boldsymbol{E}' + \alpha \nabla_t' \boldsymbol{B}' = 0$
- $\boldsymbol{\nabla} \times \boldsymbol{E} - \boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) + \alpha \nabla_t \boldsymbol{B} - \alpha (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{B} = 0$
- with $\boldsymbol{\nabla} \times (\boldsymbol{u} \times \boldsymbol{B}) = \boldsymbol{u}(\boldsymbol{\nabla} \cdot \boldsymbol{B}) - (\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{B}$ we obtain
- $\boldsymbol{\nabla}' \times \boldsymbol{E}' + \alpha \nabla_t' \boldsymbol{B}' = \boldsymbol{\nabla} \times \boldsymbol{E} + \alpha \nabla_t \boldsymbol{B} + (1 - \alpha)(\boldsymbol{u} \cdot \boldsymbol{\nabla}) \boldsymbol{B}$
- therefore $\alpha = 1$

Summary

- defining $\boldsymbol{E}$ and $\boldsymbol{B}$ by $\dot{\boldsymbol{p}} = q\{\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\}$
- superposition principle
- electric charge is conserved
- equations should be the same in all inertial frames
- even with space and time inversion
- *Make it as simple as possible, but not simpler*
- the electromagnetic field exists in empty space

Unique solution is set of Maxwell's equations

1. $\epsilon_0 \boldsymbol{\nabla} \cdot \boldsymbol{E} = \varrho$
2. $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$
3. $\boldsymbol{\nabla} \times \boldsymbol{E} + \nabla_t \boldsymbol{B} = 0$
4. $(1/\mu_0)\boldsymbol{\nabla} \times \boldsymbol{B} - \epsilon_0 \nabla_t \boldsymbol{E} = \boldsymbol{j}$