Fundamental particles

Spin and statistics

Many particles

Gibbs state

Fermions

Zero temperature

White dwarfs and neutron stars

Bosons

Black body radiation

Bose-Einstein condensation

Fermions, Bosons and their Statistics

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March/April 2011

Structure fermions

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- p is stable, n $\to$ p+e$^-$+$\bar{\nu}$ is allowed

Exchange bosons

W^-	$\bar\nu_e e^-\ \bar\nu_\mu \mu^-\ \bar\nu_\tau \tau^-$	$\bar u d\ \bar c s\ \bar t b$
γ, Z^0	$e^+ e^-\ \mu^+\mu^-\ \tau^+\tau^-\ (\bar\nu_e \nu_e\ \bar\nu_\mu \nu_\mu\ \bar\nu_\tau \nu_\tau)$	$\bar d d\ \bar u u\ \bar s s\ \bar c c\ \bar b b\ \bar t t$
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- is there also a Higgs boson H ?

The Aleph detector (opened) at Cern

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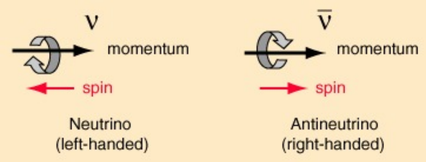
- 

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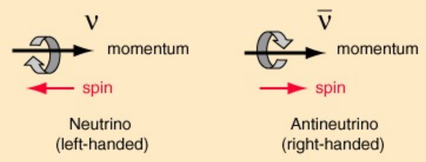
- 

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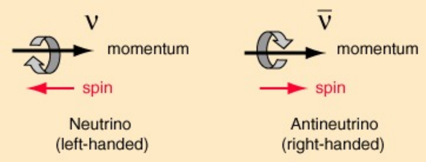
- 

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- Pauli exclusion principle

Wolfgang Pauli, Austrian/Swiss Physicist, Nobel prize 1945

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- note that the number of particles in a system is not fixed (open system)

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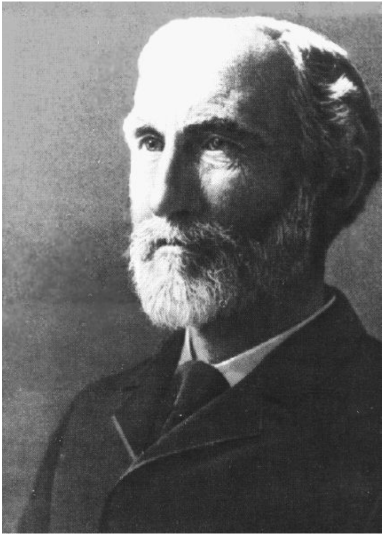
Free energy

- The equilibrium (Gibbs) state of an open system is
  $G = e^{(F - H + \mu N)/k_B T}$
- free energy F defined by $\mathrm{tr}\, G = 1$
- which gives
  $F = -k_B T \ln \mathrm{tr}\, e^{(\mu N - H)/k_B T}$
- temperature T defined by $\mathrm{tr}\, GH = U = \bar{H}$
- chemical potential μ defined by $\mathrm{tr}\, GN = \bar{N}$
- generalization to more than one species of particles is obvious



Josiah Willard Gibbs, US American physicist, 1839-1903

Fermi-Dirac statistics

- because particle number operators commute

$$e^{(\mu N - H)/k_B T} = \prod_j e^{(\mu - \epsilon_j) N_j / k_B T}$$

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$$\text{tr}\; e^{(\mu N - H)/k_B T} = \prod_j \left(1 + e^{(\mu - \epsilon_j)/k_B T}\right)$$

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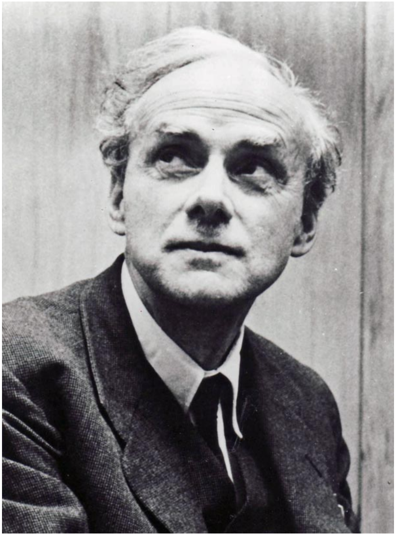
- in $[\epsilon, \epsilon + \mathrm{d}\epsilon]$ there are $V\, z(\epsilon)\, \mathrm{d}\epsilon$ single particle states
- one may write

$$F = -k_B T V \int \mathrm{d}\epsilon\; z(\epsilon) \ln \left(1 + e^{(\mu - \epsilon)/k_B T}\right)$$

Enrico Fermi, Italian/USA physicist, 1901-1954



Paul Dirac, Britisch physicist, 1902-1984

Non-relativistic fermions

- sum over states (factor 2 for spin)

$$\sum_j = 2 \int \frac{\mathrm{d}^3 x \, \mathrm{d}^3 p}{(2\pi\hbar)^3}$$

Non-relativistic fermions

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Fundamental particles

Spin and statistics

Many particles

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Zero temperature

White dwarfs and neutron stars

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Black body radiation

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Non-relativistic fermions

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$$\sum_j = 2 \int \frac{\mathrm{d}^3x\,\mathrm{d}^3p}{(2\pi\hbar)^3}$$

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- gives

$$\sum_j = V \int \mathrm{d}\epsilon\, z(\epsilon)$$

$$= V\,\frac{1}{\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int \mathrm{d}\epsilon\,\sqrt{\epsilon}$$

Particle density and pressure

- recall

$$F = -k_B T V \int d\epsilon \, z(\epsilon) \ln \left(1 + e^{(\mu - \epsilon)/k_B T} \right)$$

Particle density and pressure

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- pressure is

$$\begin{aligned} p &= -\frac{\partial F}{\partial V} \\ &= k_B T \int \mathrm{d}\epsilon\, z(\epsilon) \ln \left(1 + e^{(\mu - \epsilon)/k_B T} \right) \\ &= \frac{2}{3} \int \mathrm{d}\epsilon\, z(\epsilon)\, \frac{\epsilon}{e^{(\epsilon - \mu)/k_B T} + 1} \end{aligned}$$

Zero temperature

- if $T \to 0$
$$\frac{1}{e^{(\epsilon - \mu)/k_B T} + 1} = \theta(\mu - \epsilon)$$

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$$p(0,\mu) = \frac{2}{3}\int_{-\infty}^{\mu} d\epsilon \, z(\epsilon)\,\epsilon$$

- if particle density $\bar{n}$ is given, then

$$\bar{n} = n(0,\epsilon_F) = \int_{-\infty}^{\epsilon_F} d\epsilon \, z(\epsilon)$$

defines Fermi energy

Atoms and so forth

- Periodic system

Atoms and so forth

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- from small to huge molecules

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- dielectrics, conductors, semiconductors, ferromagnets

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- stars can be stable even at $T = 0$

Hydrodynamic equilibrium

- The mass within a sphere of radius R is
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- relate pressure with mass density

Equation of state

- electron gas at $T = 0$

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$$p = a \, \frac{\hbar^2}{m} \, n^{5/3} \text{ where } a = 1.2058$$

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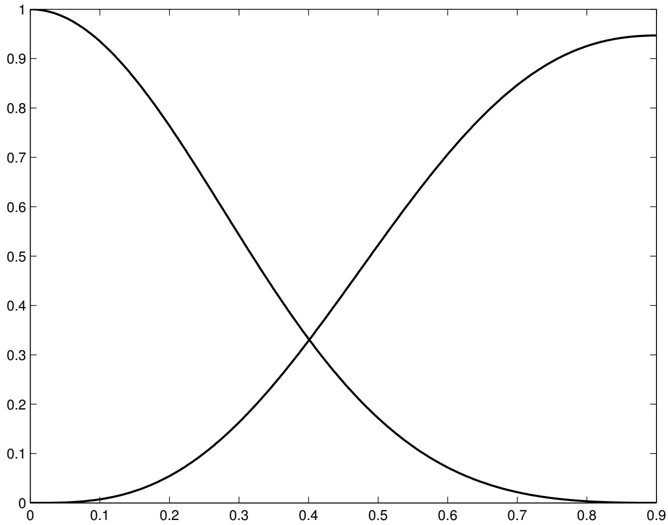
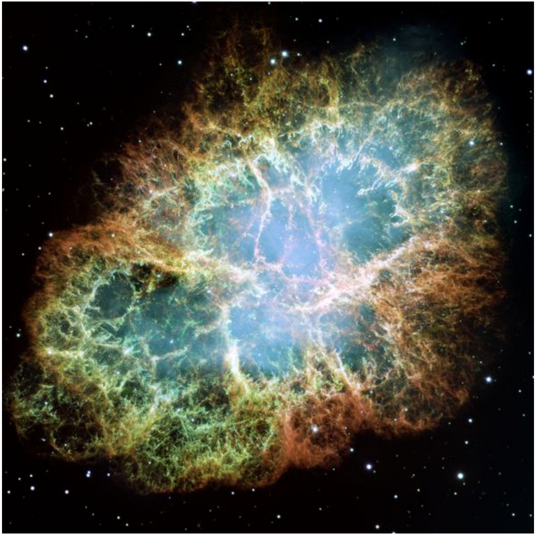
- since one electron is related with two nucleons

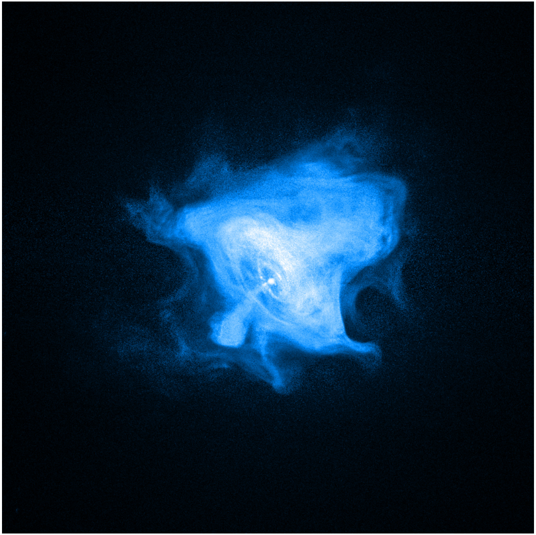
$$p = a\,\frac{\hbar^2}{m_{\rm e}}\left\{\frac{\rho}{2m_{\rm p}}\right\}^{5/3}$$



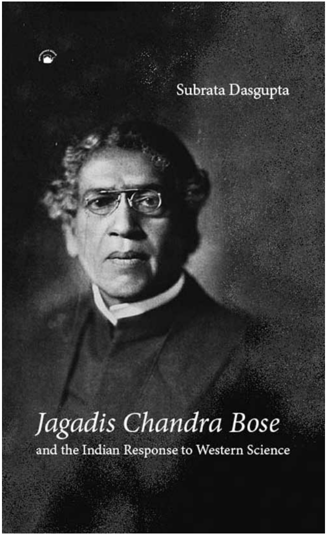
Pressure p (decreasing) and mass M (increasing) of a white dwarf vs. distance r from the center in natural units. We have assumed zero temperature, two nucleons per electron which are treated non-relativistically. $r = 1$ corresponds to 6500 km, $M = 1$ to 0.85 sun masses.



Crab nebula, the cloud of debris of the 1054 supernova



There is a neutron star (pulsar) at its center.



Jagadris Chandra Bose, Indian physicist, 1858-1937

Bosons

- bosons have integer spin or integer helicity

Fundamental particles

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- or phonons, the quanta of lattice vibrations

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Many bosons

- recall the spin/statistics theorem

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- $N = \sum_j N_j$ - total number of indistinguishable particles

Free energy

- energy is $H = \sum_j \epsilon_j A_j^* A_j + \dots$

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  $$\operatorname{tr} e^{(\mu - \epsilon_j) N_j / k_B T} = \sum_{n=0}^{\infty} \left\{ e^{(\mu - \epsilon_j)/k_B T} \right\}^n = \frac{1}{1 - e^{(\mu - \epsilon_j)/k_B T}}$$

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Free energy ctd.

- the free energy of a boson gas is

$$
\begin{aligned}
F &= k_B T \sum_j \ln\left(1 - e^{(\mu - \epsilon_j)/k_B T}\right) \\
&= k_B T \, V \int \mathrm{d}\epsilon \, z(\epsilon) \ln\left(1 - e^{(\mu - \epsilon)/k_B T}\right)
\end{aligned}
$$

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$$= k_B T\, V \int \mathrm{d}\epsilon\, z(\epsilon) \ln\left(1 - e^{(\mu - \epsilon)/k_B T}\right)$$

- compare with the free energy of a fermi gas

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- recall spin/statistics theorem: again the $+/-$ difference

Black body radiation

- photon energy is $\hbar\omega = pc$
- density of states can be read off from

$$2\int \frac{\mathrm{d}^3x\,\mathrm{d}^3p}{(2\pi\hbar)^3} = V\int \frac{\mathrm{d}\omega\,\omega^2}{\pi^2 c^3}$$

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$$F = \mathrm{k_B}T\,V\int_0^\infty \frac{\mathrm{d}\omega\,\omega^2}{\pi^2 c^3}\,\ln\left(1 - e^{-\hbar\omega/\mathrm{k_B}T}\right)$$

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- internal energy U
- $U = F + TS = F - T\partial F/\partial T$

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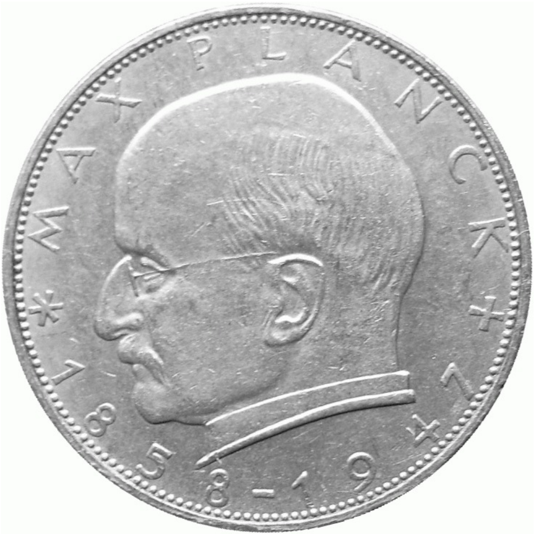
- internal energy U
- $U = F + TS = F - T\partial F/\partial T$
- Planck's formula

$$U = V\int_0^\infty \frac{\mathrm{d}\omega\,\omega^2}{\pi^2c^3}\,\frac{\hbar\omega}{e^{\hbar\omega/\mathrm{k_B}T} - 1}$$



Max Planck, German physicist, 1858-1947

Particle density

- recall

$$\sum_{n=0}^{\infty} e^{n(\mu - \epsilon_j)/\mathrm{k_B}T} = \frac{1}{1 - e^{(\mu - \epsilon_j)/\mathrm{k_B}T}}$$

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- valid only if $\mu < \min \epsilon_j = 0$

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- the average particle density $\bar{n}$ is therefore
$$\bar{n}(T,\mu) = \int_0^\infty \mathrm{d}\epsilon\, z(\epsilon)\, \frac{1}{e^{(\epsilon-\mu)/k_B T} - 1}$$

Particle density

- recall
$$\sum_{n=0}^{\infty} e^{\,n(\mu-\epsilon_j)/k_B T} = \frac{1}{1-\,e^{\,(\mu-\epsilon_j)/k_B T}}$$
- valid only if $\mu < \min \epsilon_j = 0$
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- $\bar{n}(T,0)$ is maximal density:
$$\bar{n}_{\max}(T) = \int_0^\infty \mathrm{d}\epsilon\, z(\epsilon)\,\frac{1}{e^{\,\epsilon/k_B T}-1}$$

Bose-Einstein condensation

- If T decreases, so does $\bar{n}_{\max}(T)$

Fundamental particles

Spin and statistics

Many particles

Gibbs state

Fermions

Zero temperature

White dwarfs and neutron stars

Bosons

Black body radiation

Bose-Einstein condensation

Bose-Einstein condensation

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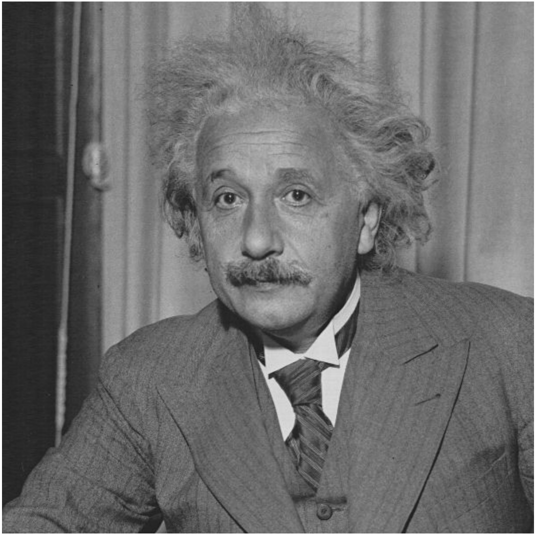

Albert Einstein, German physicist, 1879-1955