Entropy

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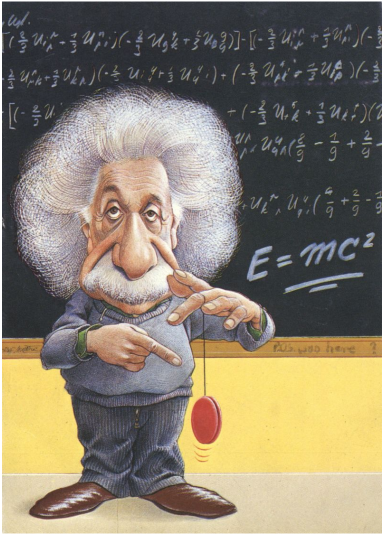
April 1, 2010



Make it as simple as possible, but not simpler.

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- i.e. all probabilities vanish besides $w_i = 1$

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- Think of an experiment with a beam of photons (light) or other particles
- The essence of quantum theory:
- Single measurements have no significance, only a long series of repeated measurements – an experiment – produces reliable results.

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- work it out for large numbers

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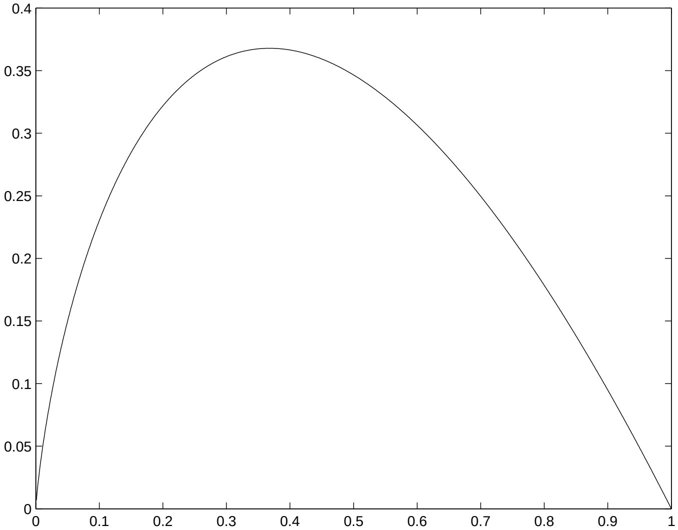

Graph of $y = -x \ln x$ between $0 \leq x \leq 1$

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- $S(U^\dagger W U) = S(U)$


REPUBLIK ÖSTERREICH
S 3
1844
1906
75. Todestag des Physikers
LUDWIG BOLTZMANN
S. BUCHNER
1981
A. NEFE

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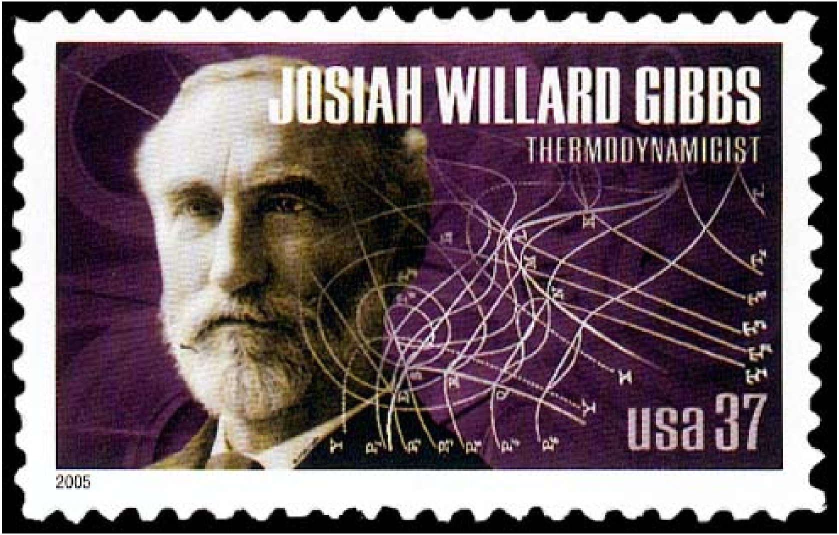
- ...until thermodynamic equilibrium has been attained
- The corresponding state G is called a Gibbs state


JOSIAH WILLARD GIBBS
THERMODYNAMICIST
usa 37
2005

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- $G = e^{(F-H)/\mathrm{k_B}T}$

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- $\operatorname{tr} H(G + \delta W) = U$
- i.e. $\operatorname{tr} H \delta W = 0$
- 2 Lagrange parameters T and F
- $\operatorname{tr} \delta W \{-k_B T \ln G + F - H\} = 0$
- $G = e^{(F - H)/k_B T}$
- $\operatorname{tr} G = 1$ implies $F = -k_B T \ln e^{-H/k_B T}$

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- $\mathrm{tr}\, \delta W = 0$
- $\mathrm{tr}\, H(G + \delta W) = U$
- i.e. $\mathrm{tr}\, H \delta W = 0$
- 2 Lagrange parameters T and F
- $\mathrm{tr}\, \delta W \{-k_B T \ln G + F - H\} = 0$
- $G = e^{(F - H)/k_B T}$
- $\mathrm{tr}\, G = 1$ implies $F = -k_B T \ln e^{-H/k_B T}$
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The Gibbs state

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- T is the temperature, $T > 0$
- $U = \mathrm{tr}\,GH$ increases with T