

Mode Coupling and Propagation

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We discuss the propagation of light in a single or in a set of coupled waveguides. The field may be expanded into guided modes which travel with different propagation constants. One may also tackle the problem directly. We derive the Fresnel equation and discuss a solution strategy in terms of the finite difference method. The boundaries of the artificial computational window lead to unwanted reflexions; we show a remedy in terms of transparent boundary conditions.

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1 TE Modes

A planar waveguide is characterized by a permittivity profile which depends on one coordinate only, say x . If $\epsilon = \epsilon(x)$ is a smoothly varying function¹ we speak of a graded index waveguide.

A slab waveguide consists of optically homogeneous slabs. The lowest is the substrate, on top of which there are one or many thin films, the topmost region is the cover. A slab waveguide is characterized by a piecewise constant permittivity profile.

We choose z as the direction of propagation. If the electric field is polarized transversally to the waveguide normal and to the direction of propagation, we speak of a TE mode. E_x and E_z vanish while E_y has the form

$$E_y(t, x, y, z) = e^{-i\omega t} e^{i\beta z} E(x) . \quad (1)$$

The amplitude $E = E(x)$ has to obey the mode equation

$$E'' + k_0^2 \epsilon(x) E = \beta^2 E . \quad (2)$$

$k_0 = \omega/c_0$ denotes the vacuum wave number and $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$ is the speed of light in vacuum. The mode equation results by inserting (2) into the second order Maxwell equation

$$\text{curl curl } \mathbf{E} = k_0^2 \epsilon(x, y, z) \mathbf{E} . \quad (3)$$

The power transported by (1) is proportional to the waveguide width and to

$$\int dx |E(x)|^2 . \quad (4)$$

Therefore, only square integrable solutions of (2) describe guided modes.

1.1 Setting up a multilayer slab waveguide

The following piece of MATLAB code sets up a multilayer slab waveguide.

```
1 function wg=setup_waveguide(wid,ndx,NX,k0)
2 % usage: wg=setup_waveguide(wid,ndx,NX,k0);
3 NS=length(wid); % number of slabs
4 pos=cumsum(wid);
5 wg.x=linspace(0,pos(NS),NX);
6 wg.eps=ndx(1)^2*(wg.x<pos(1));
7 for s=2:NS
8     wg.eps=wg.eps+ndx(s)^2*((wg.x>=pos(s-1))&(wg.x<pos(s)));
9 end;
10 wg.eps(NX)=ndx(NS)^2;
11 wg.k0=k0;
```

¹with a jump at the interface between the modified substrate and the cover

`wid` and `ndx` are vectors of equal length, they describe the width and the refractive index of consequent layers. `NX` is the number of equally spaced mesh points, `k0` the vacuum wavenumber of light by which the waveguide will be analyzed. The program returns a record with fields `x`, `eps`, and `k0`. The former describes the waveguide cross section (here: one-dimensional), the second the permittivity profile, the third is k_0 .

Here we present an example of a structure made up of a substrate, three films of increased refractive index and separated by gaps, the whole thing covered by air. This program

```

1  function wg=wg_example()
2  % usage: wg=wg_example();
3  wid=[4.0,1.5,1.0,1.5,1.0,1.5,1.5,2.0];
4  ndx=[1.49,1.52,1.49,1.52,1.49,1.52,1.49,1.00];
5  lambda=0.6328;
6  k0=2*pi/lambda;
7  wg=setup_waveguide(wid,ndx,512,k0);
8  plot(wg.x,wg.eps,'-k','linewidth',1.8);
9  print -depsc 'wg_example.eps'
10 ! epstopdf wg_example.eps
11 ! del wg_example.eps

```

generates the waveguide `wg` and represents it graphically by a PDF file. See Figure 1.

1.2 Modes

Let us now calculate the guided modes of such a multilayer waveguide. We solve the mode equation

$$\frac{1}{k_0^2} E'' + \epsilon(x) E = \bar{\epsilon} E \quad (5)$$

numerically. The effective permittivity $\epsilon_{\text{eff}} = \bar{\epsilon} = (\beta/k_0)^2$ stands for an eigenvalue of the linear operator

$$L = \frac{1}{k_0^2} \frac{d^2}{dx^2} + \epsilon(x) . \quad (6)$$

If a function $f = f(x)$ is represented by $f_j = f(x_j)$ on a mesh of equally spaced points x_j , we work out its second derivative by

$$f_j'' = f''(x_j) = \frac{f_{j-1} - 2f_j + f_{j+1}}{h^2} , \quad (7)$$

where h denotes the mesh width.

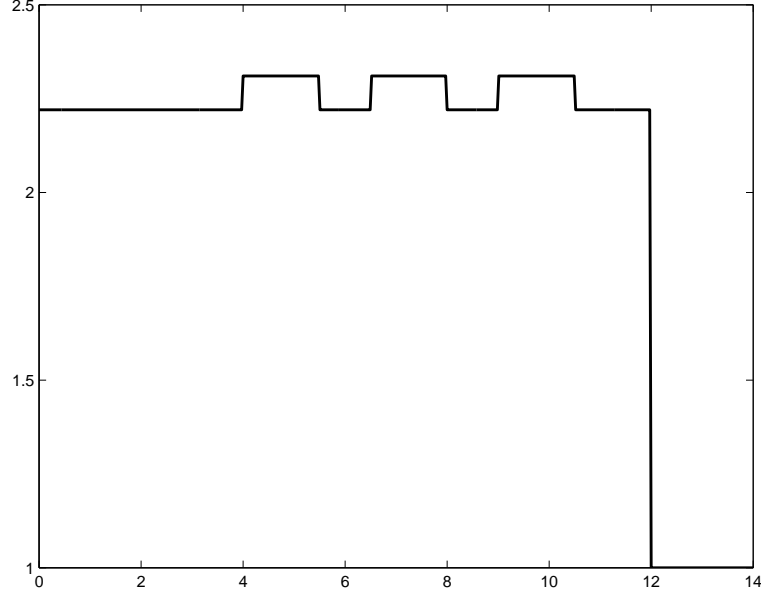


Figure 1: Permittivity $\epsilon = \epsilon(x)$ vs. cross section coordinate x . Three wave-guiding films are mounted on a substrate and separated by two gaps of substrate refractive index. The structure is covered by air.

For $j = 1, 2, \dots, N$ we set $x_j = jh$ and $E_j = E(x_j)$. Likewise, the permittivity is approximated by $\epsilon_j = \epsilon(x_j)$. We assume $E_0 = 0$ just as $E_{N+1} = 0$. Outside of our computational window there is infinity where the field vanishes. For $F = LE$ or $F_j = \sum_k L_{jk} E_k$, we obtain

$$F_1 = \frac{E_0 - 2E_1 + E_2}{k_0^2 h^2} + \epsilon_1 E_1 = \frac{-2E_1 + E_2}{k_0^2 h^2} + \epsilon_1 E_1, \quad (8)$$

$$F_2 = \frac{E_1 - 2E_2 + E_3}{k_0^2 h^2} + \epsilon_2 E_2, \quad (9)$$

and so forth, until

$$F_N = \frac{E_{N-1} - 2E_N + E_{N+1}}{k_0^2 h^2} + \epsilon_N E_N = \frac{E_{N-1} - 2E_N}{k_0^2 h^2} + \epsilon_N E_N. \quad (10)$$

From these equations the structure of the matrix $L = L_{jk}$ may be read off. Its diagonal elements are

$$L_{jj} = -\frac{2}{k_0^2 h^2} + \epsilon_j, \quad (11)$$

while its side diagonals are

$$L_{j-1,j} = L_{j,j+1} = \frac{1}{k_0^2 h^2}, \quad (12)$$

or zero, if one of the indices is not in the range from 1 to N.

The following MATLAB program calculates all modes of a waveguide, guided or not.

```

1 function [em,ev]=calculate_modes(wg)
2 % usage: [em,ev]=calculate_modes(wg);
3 NX=length(wg.x);
4 HX=wg.x(2)-wg.x(1);
5 main=-2*ones(NX,1)/HX^2/wg.k0^2+wg.eps';
6 next=1*ones(NX-1,1)/HX^2/wg.k0^2;
7 L=diag(next,-1)+diag(main,0)+diag(next,1);
8 [em,ev]=eig(L);
9 ev=diag(ev);

```

`em(:,r)`, a column vector, represents the eigenfunction labeled by r . `ev(r)` is the corresponding eigenvalue, an effective permittivity.

Recall that a mode is guided (i.e. square integrable) if its effective permittivity is larger than $\epsilon(-\infty)$ as well as larger than $\epsilon(+\infty)$. Its effective permittivity will be smaller than the maximum permittivity value.

We single out the guided modes and plot them.

```

1 function plot_guided_modes(wg)
2 % usage: plot_guided_modes(wg);
3 [em,ee]=calculate_modes(wg);
4 NX=length(wg.x);
5 epssub=wg.eps(1);
6 epscov=wg.eps(NX);
7 epslow=max([epssub,epscov]);
8 gm=em(:,ee>epslow);
9 plot(wg.x,gm,wg.x,0.1*wg.eps,'-k','linewidth',2);

```

Figure 2 shows the fields of the six guided modes.

1.3 Propagation by mode expansion

By introducing the scalar product

$$(g, f) = \int dx g^*(x) f(x) \quad (13)$$

we easily may show that the mode operator (6) is self-adjoint, or hermitian. Therefore, its eigenfunctions form a complete set of normalized and mutually

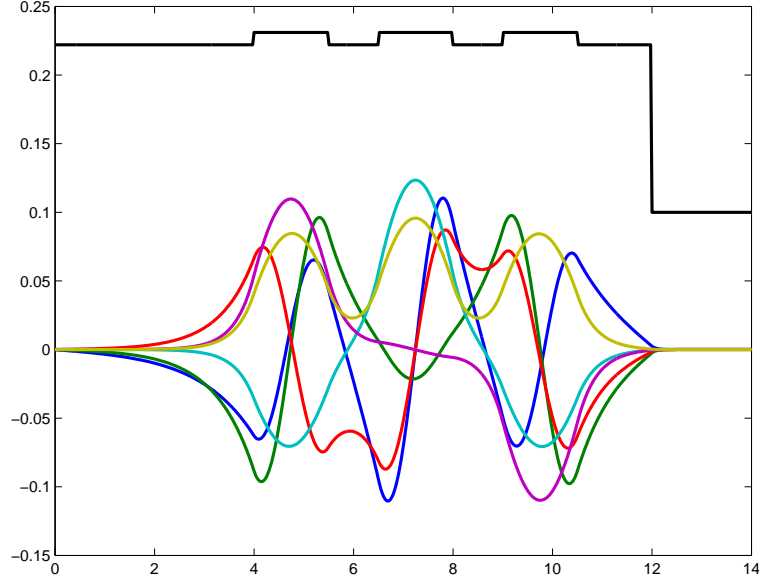


Figure 2: The waveguide structure as described above carries six guided modes. Superimposed is the waveguide profile (scaled differently). Note that the guided modes barely penetrate into the cover. However, there is some overlap in the gap regions.

orthogonal functions. Any square integrable function $f = f(x)$ may be decomposed into mode fields $e_r = e_r(x)$, i. e.

$$f(x) = \sum_r c_r e_r(x) , \quad (14)$$

and the expansion coefficients c_r are given by

$$c_r = (e_r, f) = \int dx e_r^*(x) f(x) . \quad (15)$$

If we want to know how an electric field $E = E(x, z)$ depends on the propagation coordinate z we just have to work out the expansion coefficients

$$c_r = \int dx e_r^*(x) E(x, 0) \quad (16)$$

and write

$$E(x, z) = \sum_r c_r e_r(x) e^{i\beta_r z} , \quad (17)$$

as follows from (1).

However, there is a problem. $\beta_r = \pm k_0 \sqrt{\epsilon_r}$ has two solutions, which one should be chosen? Put otherwise, should we really use all modes for expanding the initial field into eigenfunctions? We have just touched a deep problem. The differential equation for propagating a field along z is of second order, and therefore, two initial conditions should be specified, not just one, namely $f(x) = E(x, 0)$. We come back to this in the next section.

The following numerical study provides a convincing answer. We have expanded a Gaussian field profile into all modes, into modes with positive effective permittivity and into guided modes.

The field has been generated by

```
1 function f=setup_incident_beam(wg)
2 % usage f=setup_incident_beam(wg);
3 pos=6.0;
4 wid=1.0;
5 f=exp(-(wg.x-pos)/wid).^2);
```

Note that this program returns a column vector. And here is the code for expanding the field into modes.

```
1 function overlap=expand_field(wg,f)
2 % usage: overlap=expand_field(wg,f);
3 [em,ee]=calculate_modes(wg);
4 c=em'*f;
5 fa=em*c;
6 pm=em(:,ee>0);
7 c=pm'*f;
8 fp=pm*c;
9 NX=length(wg.eps);
10 epssub=wg.eps(1);
11 epscov=wg.eps(NX);
12 epslow=max([epssub,epscov]);
13 gm=em(:,ee>epslow);
14 c=gm'*f;
15 fg=gm*c;
16 overlap=abs(fg'*f)^2/(fg'*fg)/(f'*f);
17 x=wg.x;
18 plot(x,f,x,fa,x,fp,x,fg,x,wg.eps,'-k','linewidth',2);
```

Line 1 says that the waveguide **wg** is investigated. An incident field **f** is analyzed. It will be expanded by using all modes (**fa**), or by using modes with positive permittivity only (**fp**) or by using guided modes only (**fg**). Line 2 provides the eigenvectors (**em**) and its corresponding effective epsilons (**ee**). The next two lines deliver the coefficients **c** of the decomposition and its reconstruction **fa**, using all eigenvectors, i. e. **em**.

The next section uses **pm**, the set of all eigenvectors with positive permittivity.

The following section singles out guided modes as described before and delivers a decomposition of the original field into the fields of guided modes.

The final part of this program produces a PDF file which visualizes all this. See Figure 3.

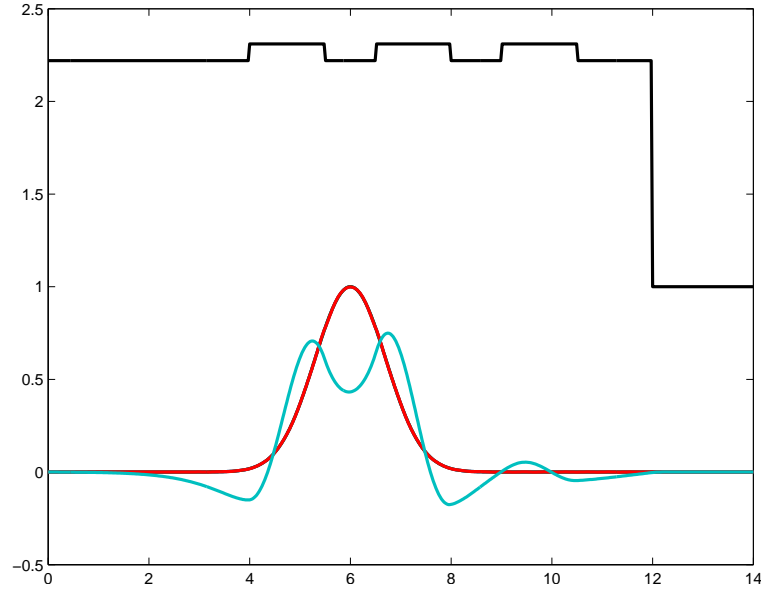


Figure 3: A Gaussian beam field is decomposed into the modes of a waveguide. The waveguide structure is superimposed in another scale. The original field and its representation by all modes are numerically identical. Using modes with positive effective permeability is a very good approximation, it cannot be distinguished from the original in this picture. Using guided modes only (the curve with dips at the barriers) evidently is not a good idea.

The conclusion is: Incident smooth fields can be approximated very well, but not identically, by a sum of modes with positive effective permittivities. In our case the overlap R between the incident beam f and its representation g by guided modes, as defined by $|(g, f)|^2 = R(g, g)(f, f)$ turned out to be 0.7482. Remember this number! We come back to it. Incidentally, the Cauchy-Schwartz inequality says $0 \leq R \leq 1$.

The following piece of code realizes the propagation and generates a contour plot. See Figure 4.

```

1 function propagate_pmodes(wg,f,dist,HZ)
2 % usage: propagate_pmodes(wg,f,dist,HZ);
3 [em,ee]=calculate_modes(wg);
4 positive=ee>0;
```

```

5  pm=em(:,positive);
6  pe=ee(positive);
7  NC=length(pe);
8  beta=wg.k0*sqrt(pe);
9  c=pm'*f;
10 NZ=dist/HZ;
11 z=linspace(0,dist,NZ);
12 cz=zeros(NC,NZ);
13 for r=1:NC
14     cz(r,:)=c(r)*exp(i*beta(r)*z);
15 end;
16 fz=pm*cz;
17 contour(z,wg.x,abs(fz).^2,20);

```

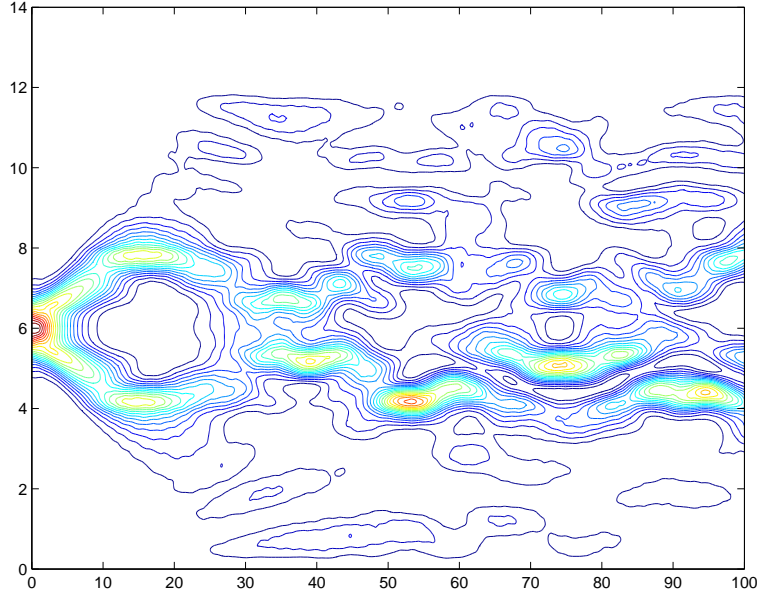


Figure 4: The field of Figure 3 was decomposed into modes with positive effective epsilon, each mode propagating along z with its appropriate propagation constant. We plot the contour lines of the power. Bottom to top: waveguide cross section x , microns. Left to right: propagation direction z , microns.

1.4 Difficulties with the mode propagation method

We have just described a seemingly simple method how to calculate the propagation of an incident beam in a waveguide structure. We decompose the incident

beam into a superposition of a reasonably chosen set of modes and let each mode propagate with its appropriate propagation constant. However, there is a certain amount of arbitrariness.

Why choose only positive real propagation constants? Clearly, because we consider the incident beam to be an incident beam. It comes from left and should only propagate towards right. Therefore, the propagation constants β_r of the modes involved should be real and positive. They are real only if we restrict ourselves to modes with positive permittivity. We will look for better arguments in the following section.

There is another obvious shortcoming of the mode propagation method.

With (17) and $(e_s, e_r) = \delta_{sr}$ we easily work out

$$\int dx |E(x, z)|^2 = \sum_r |c_r|^2 . \quad (18)$$

Power is conserved. By our approximation scheme, this applies to the power within the computational window. When calculating modes, we silently assumed that they vanish outside the computational window. Hence, waves are reflected at the windows of the computational window, and therefore, although it looks quite nice, Figure 4 may be an artifact.

2 Fresnel Approximation

We will formulate the Fresnel, or par-axial approximation in order to describe a situation where a field is nearly a plane wave. The finite difference Crank-Nicholson propagation scheme is introduced and realized. Reflexions at the boundaries of the computation window result in unphysical interference effects. This problem is solved by introducing transparent boundary conditions which are discussed in the next section.

2.1 Fresnel equation

A TE polarized wave field $E_x = E_z = 0$ and $E_y = E(x, z)$ obeys the Helmholtz equation

$$(\partial_x^2 + \partial_z^2)E + k_0^2 \epsilon(x, z)E = 0 . \quad (19)$$

This is a consequence of the second order Maxwell equation (3).

If the permittivity does not depend on z , a field of the form

$$E(x, z) = E(x)e^{i\beta z} \quad (20)$$

is a solution of (19) provided

$$E'' + k_0^2 \epsilon(x)E = \beta^2 E \quad (21)$$

is fulfilled, with a square integrable function $E = E(x)$. (21) is the mode equation which we have discussed in the previous section.

We are looking for solutions of the Helmholtz equation which behave almost like modes:

$$E(x, z) = A(x, z)e^{ik_0 n_0 z}, \quad (22)$$

where the amplitude A depends but weakly on z . n_0 is a reference index which is close to the refractive index of the wave-guiding structure. We insert this into (19) and neglect $\partial_z^2 A$ as compared with $k_0 \partial_z A$. The result is

$$-i \partial_z A = P A \text{ where } P = P(z) = \frac{\partial_x^2 + k_0^2 \{ \epsilon(x) - n_0^2 \}}{2k_0 n_0}, \quad (23)$$

the Fresnel equation. P is the propagation operator². Incidentally, the condition for A not to depend on z is the mode equation, with $\beta = k_0 n_0$.

2.2 Crank-Nicholson scheme

Let $A = A(x, z)$ be represented by $A_j^r = A(jh_x, rh_z)$ on an equally spaced x, z mesh. r and j are integers in a finite range. We want to propagate the amplitudes from r to $r + 1$. The propagation operator is represented by a matrix P_{jk} .

There are three possibilities.

2.2.1 Explicitly forward

We may write

$$A_j^{r+1} = A_j^r + i h_z \sum_k P_{jk} A_k^r. \quad (24)$$

The propagation matrix has to be evaluated at $z = rh_z$. A detailed study of this propagation scheme reveals that it is never stable. See [1] for a thorough discussion.

2.2.2 Explicitly backward, or implicitly forward

We write

$$A_j^r = A_j^{r+1} - i h_z \sum_k P_{jk} A_k^{r+1}. \quad (25)$$

The propagation matrix has to be evaluated at $z = (r + 1)h_z$. (25) amounts to solving a system of linear equations. In $A^r = (I - i h_z P) A^{r+1}$ the left hand side is known, but the vector A^{r+1} is to be worked out. This scheme is much more complicated, but always stable.

² P might depend on z , but only very slowly. There might be nonlinear effects such the the refractive index changes slightly with the light intensity.

2.2.3 Crank-Nicholson scheme

It turns out that a symmetric combination of explicitly forward and explicitly backward is always stable as well, but one order in h_z more accurate. The amplitude A is propagated explicitly forward from $z = rh_z$ to $z = (r + 1/2)h_z$. It is propagated backward from $z = (r + 1)h_z$ to $z = (r + 1/2)h_z$. Both fields at $z = (r + 1/2)h_z$ should coincide, which amounts to

$$A_j^r + \frac{i h_z}{2} \sum_k P_{jk} A_k^r = A_j^{r+1} - \frac{i h_z}{2} \sum_k P_{jk} A_k^{r+1} . \quad (26)$$

Again we must solve a system of linear equations which may formally be written as

$$A^{r+1} = (I - i h_z P/2)^{-1} (I + i h_z P/2) A^r . \quad (27)$$

2.3 Unitarity

Assume for a moment that the propagation operator P does really not depend on z . Then the Fresnel equation (23) is formally solved by

$$A(z + h) = e^{i h P} A(z) . \quad (28)$$

Now, P is a self-adjoint operator, hence $e^{i h P}$ is unitary.

Explicitly forward amounts to the approximation

$$e^{i h P} \approx I + i h P . \quad (29)$$

The right hand side is not unitary.

Explicitly backward, or implicitly forward, means the approximation

$$e^{-i h P} \approx I - i h P . \quad (30)$$

The right hand side is not unitary either.

The Crank-Nicholson scheme says

$$e^{i h P} \approx \frac{I + i h P/2}{I - i h P/2} . \quad (31)$$

The right hand side of this approximation is obviously unitary and moreover correct up to order h^2 .

2.4 Propagation in a homogeneous medium

Let us now try out the Crank-Nicholson scheme by propagating a Gaussian beam in a homogeneous medium. We choose our standard example which shall consist of the substrate only. The following code does it.

```

1 function cn_free_rbc(wg,f,dist,HZ)
2 %usage: cn_free_rbc(wg,f,dist,HZ)
3 NZ=dist/HZ;
4 z=linspace(0,dist,NZ);
5 n0=sqrt(wg.eps(1));
6 NX=length(wg.x);
7 HX=wg.x(2)-wg.x(1);
8 main=-2*ones(NX,1)/HX^2/2/wg.k0/n0;
9 next=ones(NX-1,1)/HX^2/2/wg.k0/n0;
10 P=diag(next,-1)+diag(main,0)+diag(next,1);
11 F=eye(NX)+i*HZ*P/2;
12 B=eye(NX)-i*HZ*P/2;
13 fz(:,1)=f;
14 for r=2:NZ
15     f=B\F*f;
16     fz(:,r)=f;
17 end;
18 contour(z,wg.x,abs(fz).^2,32);

```

The matrix F propagates forward by $HZ/2$ while B propagates backward. Line 15 solves the system of linear equations. See Figure 5 for the result.

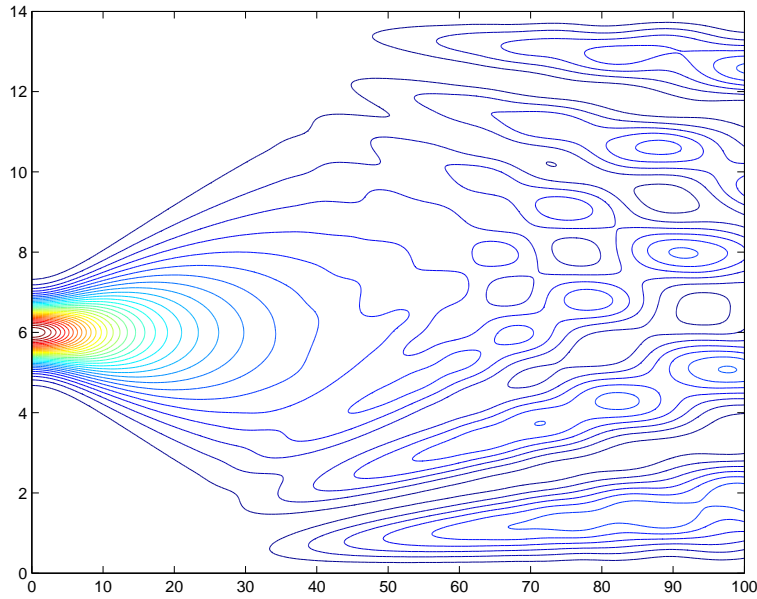


Figure 5: Propagating a Gaussian beam in a homogeneous medium by the Crank-Nicholson scheme.

Obviously, the boundaries of the computational window reflect the wave which produces unphysical interferences.

We now propagate the same beam in our example multilayer slab waveguide. Only one line (except for the figure production) has to be changed:

```
8   main=(-2*ones(NX,1)/HX^2+wg.k0^2*(wg.eps'-n0^2))/2/wg.k0/n0;
```

Figure 6 shows the result.

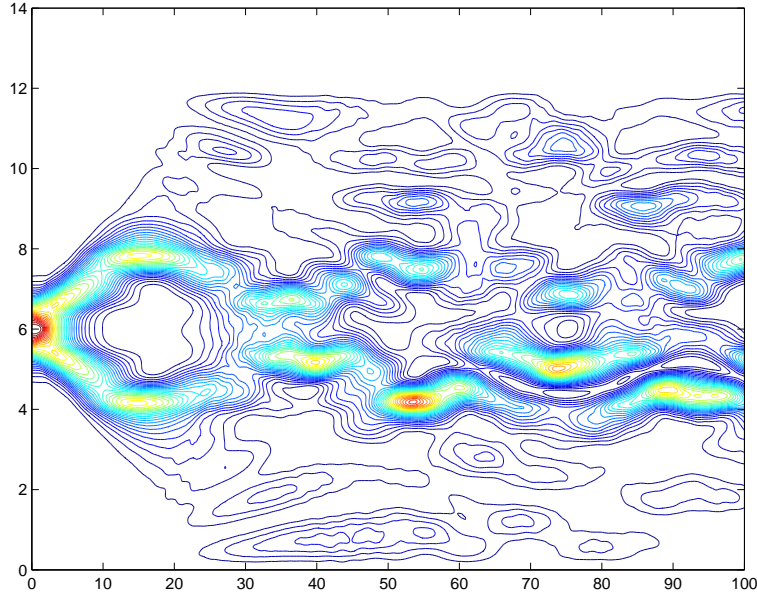


Figure 6: A Gaussian beam propagates in the multilayer slab waveguide as shown in Figure 1

Figure 6 very strongly resembles Figure 4, therefore both are wrong. When calculating modes or when setting up the Fresnel propagation operator P we have silently assumed that the field vanishes outside the computational window. This amounts to reflecting boundary conditions. What we need is a transparent computational window.

3 Transparent Boundary Conditions

We shall describe Headley's procedure how to cope with the problem of reflexions at the boundaries of the computational window. This method is tested for propagation in free space and in a multilayer waveguide structure.

3.1 Field behavior at the boundary

It is assumed that the computational window is large enough. Therefore, the field there should be an outgoing wave proportional to e^{ikx} , with a certain wavenumber. From x to $x + h_x$ there should be a phase shift by e^{ikh_x} . This means

$$e^{ikh_x} = \frac{A_{j+1}^r}{A_j^r} \quad (32)$$

close to the boundary.

Hadley's idea [2] was to work out k as

$$k = -\frac{i}{h_x} \{\log(A_N^r) - \log(A_{N-1}^r)\} . \quad (33)$$

Here N is the index of the uppermost point in the computational window.

Now, if the real part of k is positive, we have indeed an outbound wave. If, however, the real part of k turns out to be negative, we have an inbound wave. In this case we set the real part of k to zero thereby stopping reflexion. After this we may work out

$$A_{N+1}^r = e^{ikh_x} A_N^r . \quad (34)$$

And this is precisely what we need for working out the second derive at the boundary:

$$(A'')_N^r = \frac{(e^{ikh_x} - 2)A_N^r + A_{N-1}^r}{h_x^2} . \quad (35)$$

Similar considerations apply to the lower boundary.

3.2 Propagation in a homogeneous medium

Let us test Hadley's scheme of transparent boundary conditions by propagating a Gaussian beam in a homogeneous medium. We chose the example waveguide `wg` with `wg.eps` replaced by the substrate value.

```
1 function cn_free_tbc(wg,f,dist,HZ)
2 % usage: cn_free_tbc(wg,f,dist,HZ)
```

The first line says that we want to propagate, by the Crank-Nicholson scheme, in free space with transparent boundary conditions, in a waveguide `wg`, a field `f`, up to a certain distance `dist`, in steps of `HZ`. `wg` is not really required, but we need its x axis information, its substrate refractive index as well as the light wave number k_0 . The second line is a comment. It appears if you type

```
>> help cn_free_tbc
```

in the MATLAB command window.

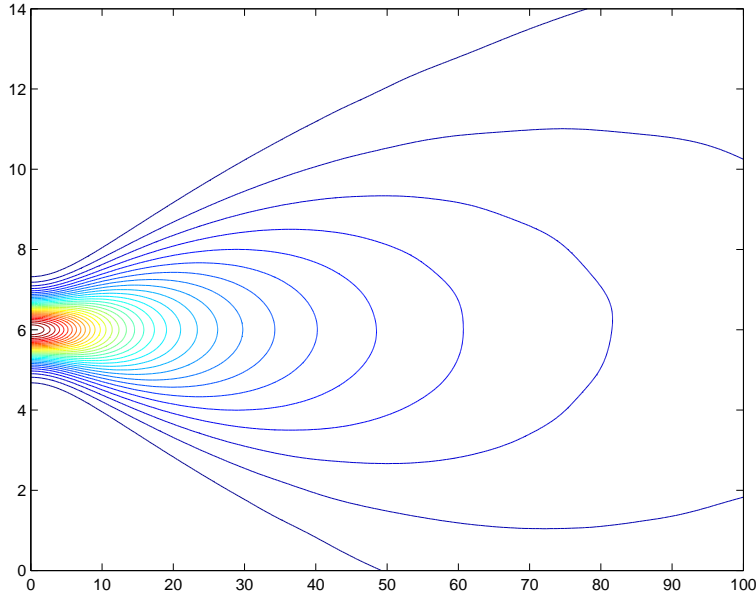


Figure 7: A Gaussian beam is inserted into a homogeneous medium. Its propagation was simulated by the Crank-Nicholson scheme for solving the Fresnel equation with Hadley's transparent boundary conditions.

```

3  TINY=1e-5;
4  NZ=dist/HZ;
5  z=linspace(0,dist,NZ);
6  n0=sqrt(wg.eps(1));
7  NX=length(wg.x);
8  HX=wg.x(2)-wg.x(1);

```

These lines define what is small, they define the z -axis, set the reference index n_0 and provide data for the computational window.

Next we set up the forward and backward propagation matrices before corrections for transparent boundary conditions.

```

9  main=-2*ones(NX,1)/HX^2/(2*wg.k0*n0);
10 next=ones(NX-1,1)/HX^2/(2*wg.k0*n0);
11 P=diag(next,-1)+diag(main,0)+diag(next,1);
12 F=eye(NX)+i*HZ*P/2;
13 B=eye(NX)-i*HZ*P/2;

```

Now we iterate:

```

14  fz(:,1)=f;

```

```

15   for r=2:NZ
16       FF=F;
17       BB=B;

```

We copy the standard forward and backward propagation matrices. They are now modified at the lower boundary

```

18       if abs(f(1))>TINY
19           k=i/HX*log(f(2)/f(1));
20           if real(k)<0
21               k=i*imag(k);
22           end;
23           tbc=exp(i*k*HX)/HX^2*i*HZ/2/(2*wg.k0*n0);
24           FF(1,1)=FF(1,1)+tbc;
25           BB(1,1)=BB(1,1)-tbc;
26       end;

```

and at the upper boundary

```

27       if abs(f(NX))>TINY
28           k=-i/HX*log(f(NX)/f(NX-1));
29           if real(k)<0
30               k=i*imag(k);
31           end;
32           tbc=exp(i*k*HX)/HX^2*i*HZ/2/(2*wg.k0*n0);
33           FF(NX,NX)=FF(NX,NX)+tbc;
34           BB(NX,NX)=BB(NX,NX)-tbc;
35       end;

```

We now solve $BB*ff=FF*f$ and set $f=ff$.

```

36       f=BB\ (FF*f);
37       fz(:,r)=f;
38   end;

```

The next line produces Figure 7:

```

39   contour(z, wg.x, abs(fz).^2, 32);

```

3.3 Beam propagation in a multilayer slab waveguide

We have discussed the propagation of a beam in a waveguide structure before. We expanded the incident beam into modes and propagated each mode with its appropriate propagation constant. We now know that the result cannot be correct since it suffers from reflecting boundary conditions.

We have to modify just one line of the above program. Line 9 must be changed into

```
9  main=(-2*ones(NX,1)/HX^2+wg.k0^2*(wg.eps'-n0^2))/(2*wg.k0*n0);
```

The modifications to the propagation matrices are not affected by this change. Figure 8 shows the result.

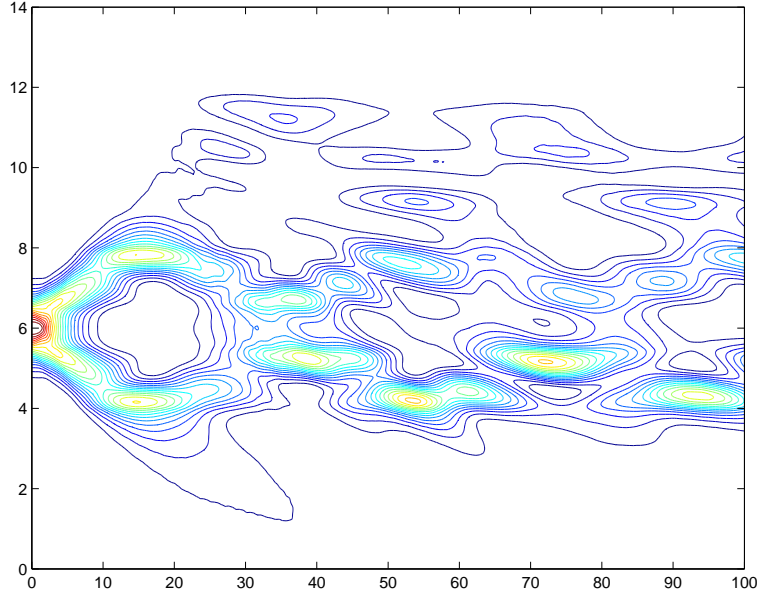


Figure 8: An incident Gaussian beam propagates along a multilayer slab waveguide. We have plotted the power $P(x, z)$ where x runs from bottom to top and z , the propagation axis, from left to right.

We conclude this tutorial article by a long distance propagation.

In Figure 9 one clearly recognizes coupling between the waveguiding layers.

Figure 10 depicts the distribution of power in the various layers.

It is evident that power has left the computational window which means that boundaries are transparent indeed.

3.4 Mode propagation reconsidered

The power in the computational window, after propagation by 1200 microns, is 0.7493 of the original power. Recall that the overlap of the incident beam with its representation by guided modes only was $R = 0.7482$. This is no coincidence. We interpret this as follows.

Everything in the incident field which is not a superposition of guided modes is radiated off. Although 1200 microns is large as compared with 14 microns for the computational window, still not all misfitting parts of the incident beam, but nearly all, have left the computational window. But as you may guess by

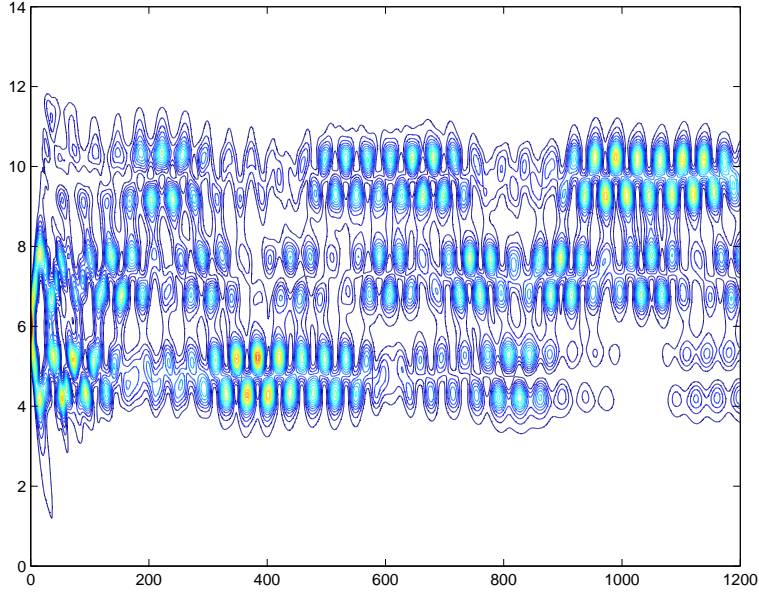


Figure 9: Propagation in a multilayer waveguide over a long distance. Distances in microns.

inspecting Figure 10, the power within the computational window will converge towards a limit, and this will be the initial overlap R , within computational accuracy.

Therefore, for understanding the significance of Figure 9, we compare it with a propagation study which relies on the decomposition of the incident beam into guided modes only. We have rejected this earlier because it could not describe the initial few microns and because it could not radiate off. But it should reflect the stationary state.

Here is the code:

```

1  function propagate_gmodes(wg,f,dist,HZ)
2  % usage: propagate_gmodes(wg,f,dist,HZ);
3  [em,ee]=calculate_modes(wg);
4  NX=length(wg.eps);
5  epssub=wg.eps(1);
6  epscov=wg.eps(NX);
7  epslow=max([epssub,epscov]);
8  guided=ee>epslow;
9  gm=em(:,guided);
10 ge=ee(guided);
11 NC=length(ge);
12 beta=wg.k0*sqrt(ge);

```

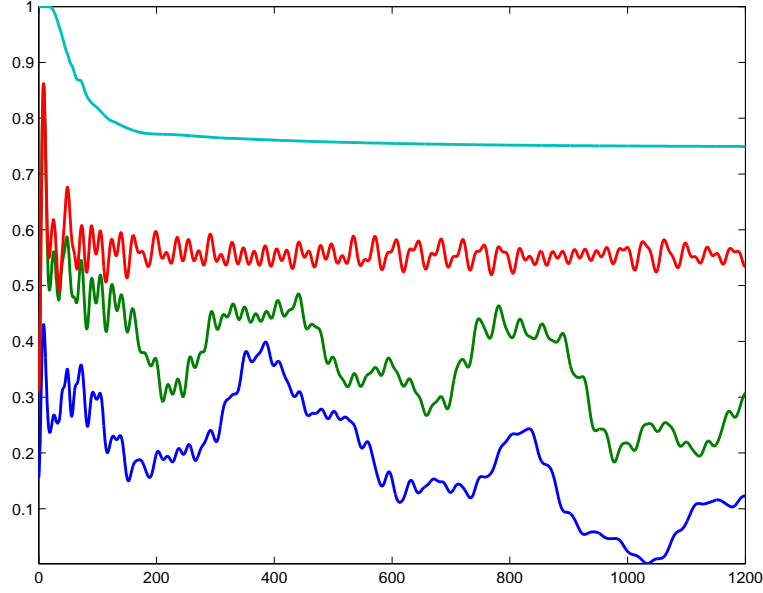


Figure 10: From bottom to top: power in the lowest guiding layer, added to it the power in the next guiding layer, than the power in all three guiding layers. The upper curve is the total power within the computational window. The difference between the two topmost curves is the power in the gaps.

```

13  c=gm'*f;
14  NZ=dist/HZ;
15  z=linspace(0,dist,NZ);
16  cz=zeros(NC,NZ);
17  for r=1:NC
18      cz(r,:)=c(r)*exp(i*beta(r)*z);
19  end;
20  fz=gm*cz;
21  contour(z,wg.x,abs(fz).^2,20);

```

You should compare this with Figure 9. Close to beam injection, at $z = 0$, the intensity patterns differ substantially. However, after a long distance of propagation, the Crank-Nicholson propagation scheme with Hadley's transparent boundary conditions cannot be discerned from a simple-minded mode propagation scheme. The incident beam field is decomposed into guided modes, and each guided mode propagates with its own propagation constant.

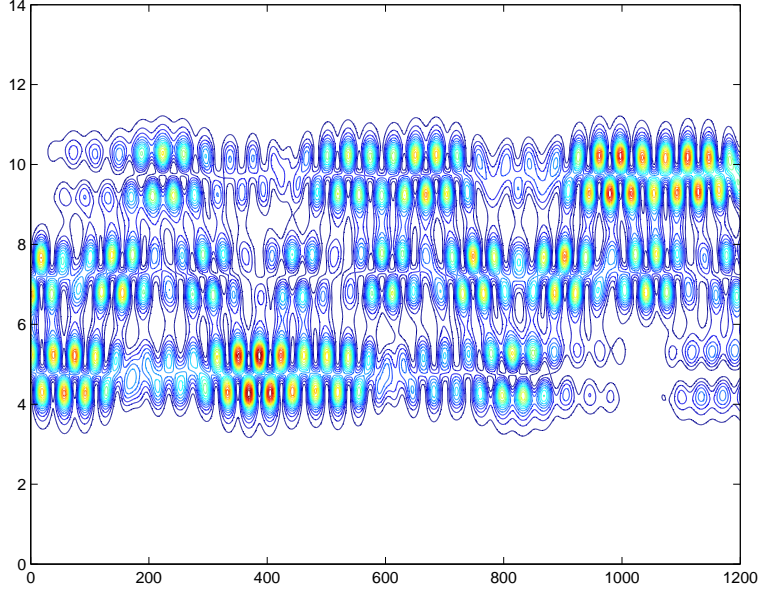


Figure 11: An incident Gaussian beam is decomposed into guided modes each of which propagates with its own propagation constant. We depict the absolute square of the field (its intensity) within the computational window. Bottom to top: computational window. Left to right: propagation distance.

4 Summary

The guided modes of multilayer slab waveguides can be very easily worked out numerically. In fact, one calculates all modes and selects those with effective permittivity larger than the maximal permittivity at infinity.

A crude approach to the propagation problem is to expand the incident beam field into all modes (we label them by r), each mode field $e_r = e_r(x)$ propagating with its own propagation constant β_r . However, the mode equation provides $e_r(x)$ and β_r^2 . Which of the two roots should be chosen?

Since the beam to be propagated is incident, only positive β_r are welcome, and therefore we restrict our attention on modes with positive β^2 , which we call positive modes. An initial Gaussian beam, for example, can be well expanded into positive modes, and the propagation as a sum of such modes seems very satisfactory, until the field hits the boundaries of the computational window. It is evident that it is reflected there which leads to unphysical interference phenomena.

We next studied the Fresnel approximation which formulates the expectation that an incident wave, from left to right, propagates almost like a plain wave

with a phase $e^{ik_0 n_0 z}$, where its amplitude $A = A(x, z)$ depends but weakly on z . The Fresnel equation is of first order with respect to the propagation variable z . Guided modes are solutions of the Fresnel equation if we insist on $\partial_z A(x, z) = 0$, with $n_0 = \beta/k_0$.

We discuss three propagation schemes. The simplest, explicitly forward, is not stable. Implicitly forward, or explicitly backward is stable, but numerically expensive because for each step a system of linear equations has to be solved. The preferred method is a combination of both, the Crank-Nicholson scheme. It is stable, and one order more accurate in the propagation step width.

However, also the Crank-Nicholson scheme propagates with a unitary propagation operator such that the power within the computational window is conserved.

Therefore we supplement the Crank-Nicholson scheme of propagation by transparent boundary conditions. At the boundary, we assume the field a plain wave with a definite wave number k . If it is an out-bound wave, nothing is done. If it is an inbound wave, we modify k such that it is neither outbound nor inbound. Thereby the field just outside the computational window may be predicted, so that we can safely calculate the second derivative at the boundary. We no longer silently assume that the field vanishes outside the computation window.

We contrast the new propagation scheme – Crank-Nicholson with transparent boundary conditions – with its counterpart for reflecting boundary conditions. The propagation of a Gaussian beam in a homogeneous medium as well as in a multilayer waveguide structure are now what we expect.

A long distance propagation by the Crank-Nicholson scheme with transparent boundary conditions is presented. Asymptotically it coincides with a propagation scheme by which only the guided modes within the incident beam propagate with their appropriate propagation constant. The former radiates off all non-mode-contributions when propagating, the latter does this immediately.

References

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- [2] G. R. Hadley, *Transparent boundary condition for beam propagation*, Opt. Lett. 16, 624-626 (1991)