Preventing waves from spreading

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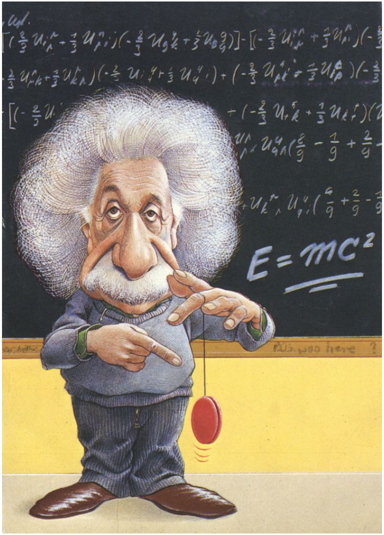

Make it as simple as possible, but not simpler.

Plane waves

- $f(t,\boldsymbol{x}) \propto \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\,\mathrm{e}^{-\mathrm{i}\omega t}$
- wave equation yields $\omega = \omega(\boldsymbol{k})$
- sound in air: $\omega = v\,|\boldsymbol{k}|$
- matter waves (particles): $\omega = \frac{\hbar}{2m}\,|\boldsymbol{k}|^2$
- light in free space: $\omega = c\,|\boldsymbol{k}|$

Wave packets

- Plane wave is an idealization
- Superposition of plane waves, i. e. wave packets
- $f(t, \boldsymbol{x}) = \int \frac{\mathrm{d}^3 k}{(2\pi)^3}\, \phi(\boldsymbol{k})\, \mathrm{e}^{\,\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \mathrm{e}^{\,-\mathrm{i}\omega(\boldsymbol{k})t}$
- $\int \mathrm{d}^3 x |f(t, \boldsymbol{x})|^2 = \int \frac{\mathrm{d}^3 k}{(2\pi)^3} |\phi(\boldsymbol{k})|^2$
- Integral over $|f(t, \boldsymbol{x})|^2$ does not depend on time
- We normalize it to 1

Location of the wave packet

- $\langle\, \boldsymbol{X}\,\rangle_t = \int \mathrm{d}^3x\, \boldsymbol{x}\, |f(t, \boldsymbol{x})|^2 =$
- $\int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \mathrm{e}^{\,\mathrm{i}\omega t}\, \phi^*(\boldsymbol{k})\, \mathrm{i}\boldsymbol{\nabla}_k \phi(\boldsymbol{k})\, \mathrm{e}^{\,-\mathrm{i}\omega t} =$
- $\int \frac{\mathrm{d}^3k}{(2\pi)^3} \phi^*(\boldsymbol{k})\, \mathrm{i}\boldsymbol{\nabla}_k \phi(\boldsymbol{k}) +$
- $t \int \frac{\mathrm{d}^3k}{(2\pi)^3} |\phi(\boldsymbol{k})|^2 \boldsymbol{\nabla}_k \omega(\boldsymbol{k})$
- $\langle\, \boldsymbol{X}\,\rangle_t = \langle\, \boldsymbol{X}\,\rangle_0 + \boldsymbol{v}\, t$
- $\boldsymbol{v} = \langle\!\langle \boldsymbol{\nabla}\omega \rangle\!\rangle = \int \frac{\mathrm{d}^3k}{(2\pi)^3} |\phi(\boldsymbol{k})|^2 \boldsymbol{\nabla}_k \omega(\boldsymbol{k})$
- group velocity

Spread of the wave packet

- $\langle\, \boldsymbol{X}^2 \,\rangle_t = \int \mathrm{d}^3x\, \boldsymbol{x}^2\, |f(t, \boldsymbol{x})|^2$
- spread $\delta X(t) = \sqrt{\langle\, \boldsymbol{X}^2 \,\rangle_t - \langle\, \boldsymbol{X} \,\rangle_t^{\,2}}$
- by a similar calculation as before:
- $\langle\, \boldsymbol{X}^2 \,\rangle_t = \dots + t^2 \,\langle\!\langle (\boldsymbol{\nabla}\omega)^2 \rangle\!\rangle$
- for large times t the spread grows as
- $\delta X(t) = |t| \sqrt{\langle\!\langle (\boldsymbol{\nabla}\omega)^2 \rangle\!\rangle - \langle\!\langle \boldsymbol{\nabla}\omega \rangle\!\rangle^2}$
- the argument of the square root cannot be negative
- Wave packets finally spread out...
- ... if the medium is homogeneous .

Electromagnetic field

- The electromagnetic field is defined by its action on charged particles
- $\dot{\boldsymbol{p}} = q\{\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\}$
- location $\boldsymbol{x}$, velocity $\boldsymbol{v}$, momentum $\boldsymbol{p}$
- charge q, electric field strength $\boldsymbol{E}$, magnetic induction $\boldsymbol{B}$
- The electromagnetic field is generated by a distribution of charged particles
- charge density ρ, current density $\boldsymbol{j}$
- Maxwell's equations
- $\mathbf{div}\,\boldsymbol{D} = \rho$, $\mathbf{div}\,\boldsymbol{B} = 0$
- $\mathbf{curl}\,\boldsymbol{H} = \boldsymbol{j} + \dot{\boldsymbol{D}}$, $\mathbf{curl}\,\boldsymbol{E} = -\dot{\boldsymbol{B}}$
- linear Medium: $\boldsymbol{D} = \epsilon\epsilon_0\boldsymbol{E}$, $\boldsymbol{B} = \mu\mu_0\boldsymbol{H}$

James Clerk Maxwell, 1831-1879

Optics

- no charges, no currents: $\rho = 0$, $\boldsymbol{j} = 0$
- no magnetic properties: $\mu = 1$
- need to study fields $\propto \mathrm{e}^{-\mathrm{i}\omega t}$ only
- $\boldsymbol{\nabla}\,\epsilon\,\boldsymbol{E} = 0$, $\mathbf{div}\,\boldsymbol{H} = 0$
- $\mathbf{curl}\,\boldsymbol{H} = -\mathrm{i}\omega\epsilon_0\,\epsilon\,\boldsymbol{E}$, $\mathbf{curl}\,\boldsymbol{E} = \mathrm{i}\omega\mu_0\boldsymbol{H}$
- With $\epsilon_0\mu_0 c^2 = 1$ and $k_0 = \omega/c$:
- $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} = k_0^2\,\epsilon\,\boldsymbol{E}$
- equivalent
- $\mathbf{curl}\,\epsilon^{-1}\,\mathbf{curl}\,\boldsymbol{H} = k_0^2\boldsymbol{H}$
- $\epsilon\,\boldsymbol{E}$ and $\boldsymbol{H}$ are automatically divergence free

Waveguides

- Spreading of light is unavoidable if the medium is homogeneous
- Therefore, the medium must be inhomogeneous if light is to be guided
- permittivity profile $\epsilon = \epsilon(\boldsymbol{x})$
- Non-constant imaginary part: microwave guides, coaxial cables
- ϵ real and non-constant: dielectric waveguides