Coupled Mode Theory

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Overview

- Mode equation
- Helmholtz equation
- Hilbert space
- Coupled waveguides
- Coupled modes
- Random waveguide array

Mode equation

- $\boldsymbol{E}(t,x,y,z) = E(x,y)\,\mathrm{e}^{\mathrm{i}\beta z}\,\mathrm{e}^{-\mathrm{i}\omega t}$
- $k_0 = \omega/c$ vacuum wave number
- propagation constant β
- general mode equation

  $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} = k_0^2\,\epsilon(x,y)\,\boldsymbol{E}$
- the curl operator is

$$\begin{pmatrix} 0 & -\mathrm{i}\beta & \partial_y \\ \mathrm{i}\beta & 0 & -\partial_x \\ -\partial_y & \partial_x & 0 \end{pmatrix}$$

- apply it twice

$$\begin{pmatrix} \beta^2-\partial_y^2 & \partial_x\partial_y & \mathrm{i}\beta\partial_x \\ \partial_x\partial_y & \beta^2-\partial_x^2 & \mathrm{i}\beta\partial_y \\ \mathrm{i}\beta\partial_x & \mathrm{i}\beta\partial_y & -\partial_x^2-\partial_y^2 \end{pmatrix}\boldsymbol{E} = k_0^2\,\epsilon(x,y)\,\boldsymbol{E}$$

Eliminate E_z

- problem: β and β^2
- problem: two polarization states, three fields
- divergence of $\epsilon\,\boldsymbol{E}$ vanishes
- $-\mathrm{i}\beta E_z = \epsilon^{-1}\partial_x \epsilon E_x + \epsilon^{-1}\partial_y \epsilon E_y$
- now the mode equation contains only two fields

$$\begin{pmatrix} k_0^2\epsilon + \partial_x \epsilon^{-1}\partial_x \epsilon + \partial_y^2 & \partial_x \epsilon^{-1}\partial_y \epsilon - \partial_x \partial_y \\ \partial_y \epsilon^{-1}\partial_x \epsilon - \partial_y \partial_x & k_0^2\epsilon + \partial_x^2 + \partial_y \epsilon^{-1}\partial_y \epsilon \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = \beta^2 \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

- and it is a normal eigenvalue problem!
- analogous form for magnetic fields

Quasi TE modes

- If waveguides are broad: $\partial_y \epsilon \approx \epsilon \partial_y$
- $E_x \approx 0$
- this results in the quasi TE mode equation
  $\{\partial_x^2 + \partial_y^2 + k_0^2\, \epsilon(x,y)\} E_y = \beta^2\, E_y$
- Helmholtz equation
- with $\partial_y \epsilon \approx \epsilon \partial_y$ and $H_x \approx 0$
- quasi TM mode equation
  $\{\epsilon \partial_x \epsilon^{-1} \partial_x + \partial_y^2 + k_0^2\, \epsilon(x,y)\} H_y = \beta^2 H_y$
- only change is $\epsilon = \epsilon(x,y)$ and additional ∂_y^2
- and: quasi modes have z-components

Hermann von Helmholtz, German physicist, 1821 - 1894; Königsberg, Bonn, Heidelberg, Berlin

Hilbert space

- Henceforth we speak about quasi-TE mode
- i. e. there is just one field component $E = E(x, y)$, the 'electric field' or the 'field', for short
- fields can be linearly combined, they form a linear space
- Power is
  $$P = \frac{2\beta}{\omega\mu_0} \int \mathrm{d}x\,\mathrm{d}y\,|E(x,y)|^2$$
- scalar product $(G, F) = \int \mathrm{d}x\,\mathrm{d}y\,G^*(x,y)\,F(x,y)$
- With this, the linear space of fields $E = E(x, y)$ with finite power transfer becomes a Hilbert space
- The Helholtz operator $H = \partial_x^2 + \partial_y^2 + k_0^2\epsilon(x, y)$
- is self-adjoint:
- $(G, HF) = (HG, F)$

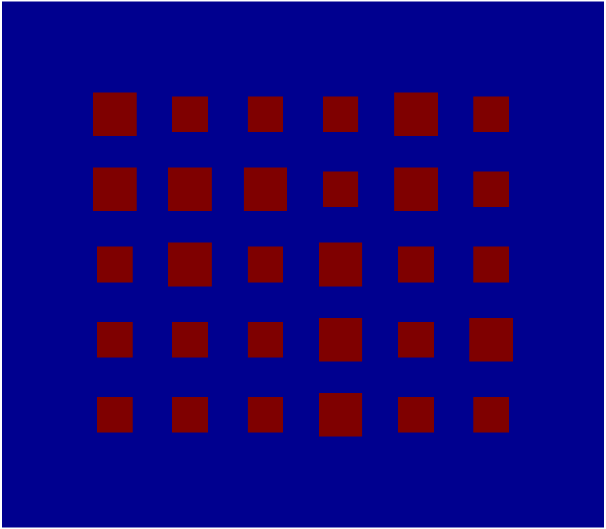
David Hilbert, German mathematician, 1862-1943; Königsberg, Göttingen

Helmholtz operator

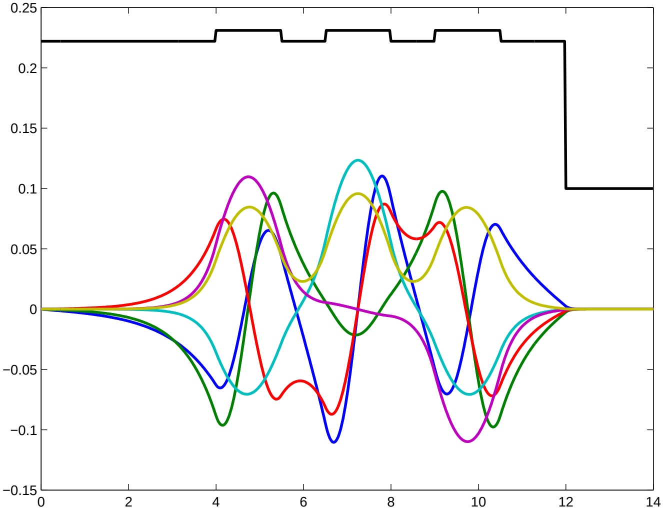
- Self adjoint operators A have remarkable properties
- $A\chi = a\chi$ guaranties that the eigenvalue a is real
- Denote by $\chi_1, \chi_2, \dots$ the normalized eigenvectors
- they form a Complete OrthoNormal Set (CONS)
- meaning $(\chi_k, \chi_j) = \delta_{jk}$
- and $\chi = \sum_j (\chi_j, \chi)\chi_j$ for all χ
- $HE = \Lambda E$ guarantees that Λ is real
- Usually, there are only a few modes E_n with positive $\Lambda_n = \beta_n^2$
- They cannot span the entire Hilbert space
- We should add wave packets of evanescent and radiation modes

Coupled waveguides

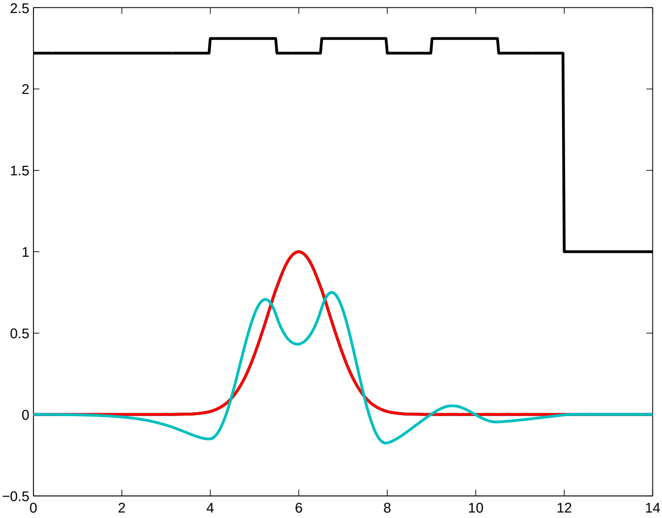
- Consider $r = 1, 2, \ldots, N$ individual waveguides
- Such as a coupler or a waveguide array
- The entire system is again a many mode waveguide
- Its modes are supermodes
- If the single waveguides are well separated
- the supermodes are given by E_r with propagation constants β_r
- However, if E_r and E_s overlap, this will no longer be true



A random waveguide array



Supermodes



Mode expansion

Coupled modes

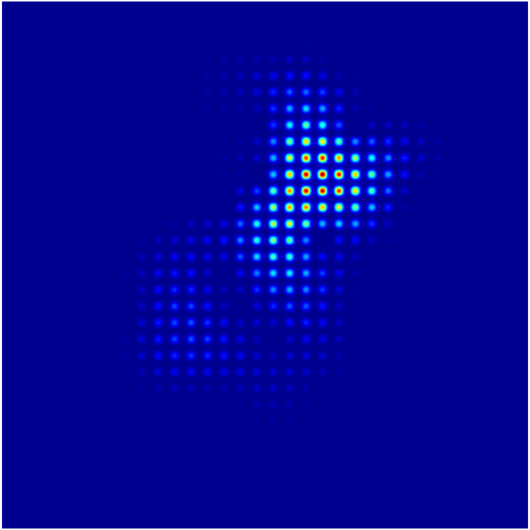
- If waveguide r is the only one, it obeys
  $(\frac{1}{k_0^2}\Delta + \epsilon_r)E_r = n_r^2 E_r$
- Supermode is described by
  $(\frac{1}{k_0^2}\Delta + \epsilon)E = n^2 E$ where $\epsilon(x,y) = \sum_r \epsilon_r(x,y)$
- bold approximation:
  $E(x,y) = \sum_r U_r E_r(x,y)$
- With
  $M_{sr} = (E_s, (\frac{1}{k_0^2}\Delta + \epsilon)E_r)$ and $\Lambda_{sr} = (E_s, E_r)$
- Solve generalized eigenvalue problem
  $M\,U = \Lambda\,U$

Coupled modes ctd.

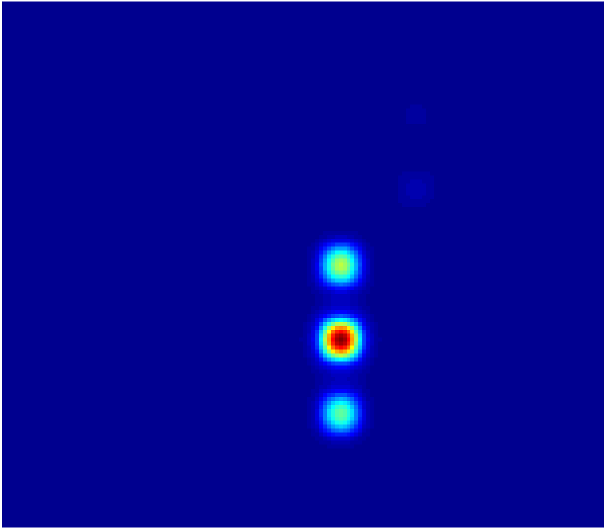
- Recall
  $M_{sr} = (E_s, (\frac{1}{k_0^2}\Delta + \epsilon)E_r)$ and $\Lambda_{sr} = (E_s, E_r)$
- Because of $(E_r, E_r) = |E_r|^2 = 1$, all diagonal elements of Λ are ones.
- However, there are non-diagonal contributions (overlaps)
- For a certain r one may write $\epsilon = \epsilon_r + \bar{\epsilon}_r$
- where $\bar{\epsilon}_r$ is the permittivity profile outside waveguide r
- such that we may write
  $M_{sr} = n_r^2 \Lambda_{sr} + (E_s, \bar{\epsilon}_r E_r)$

Random waveguide array

- `RA=rwga_descriptor()`
- `RA=rwga_single(RA)`
- `RA=rwga_overlap(RA)`
- `RA=rwga_dices(RA)`
- `RA=rwga_super(RA)`
- `RA=rwga_intensity(RA,MN)`
- Anderson localization



Ground mode of a 30×30 random waveguide array. Probability for small core is 0.1.



Finite Difference Method for a super mode.