Coupled Mode Theory

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• Mode equation
• Helmholtz equation
• Hilbert space
• Coupled waveguides
• Coupled modes
• Random waveguide array
- \( \mathbf{E}(t, x, y, z) = \mathbf{E}(x, y) e^{i\beta z} e^{-i\omega t} \)
- \( k_0 = \omega / c \) vacuum wave number
- propagation constant \( \beta \)
- general mode equation
  \[
  \text{curl curl } \mathbf{E} = k_0^2 \epsilon(x, y) \mathbf{E}
  \]
- the curl operator is
  \[
  \begin{pmatrix}
  0 & -i\beta & \partial_y \\
  i\beta & 0 & -\partial_x \\
  -\partial_y & \partial_x & 0
  \end{pmatrix}
  \]
- apply it twice
  \[
  \begin{pmatrix}
  \beta^2 - \partial^2_y & \partial_x \partial_y & i\beta \partial_x \\
  \partial_x \partial_y & \beta^2 - \partial^2_x & i\beta \partial_y \\
  i\beta \partial_x & i\beta \partial_y & -\partial^2_x - \partial^2_y
  \end{pmatrix}
  \mathbf{E} = k_0^2 \epsilon(x, y) \mathbf{E}
  \]
• problem: $\beta$ and $\beta^2$
• problem: two polarization states, three fields
• divergence of $\epsilon E$ vanishes
• $-i\beta E_z = \epsilon^{-1}\partial_x \epsilon E_x + \epsilon^{-1}\partial_y \epsilon E_y$
• now the mode equation contains only two fields

\[
\begin{pmatrix}
k_0^2\epsilon + \partial_x \epsilon^{-1}\partial_x \epsilon + \partial_y^2 & \partial_x \epsilon^{-1}\partial_y \epsilon - \partial_x \partial_y \\
\partial_y \epsilon^{-1}\partial_x \epsilon - \partial_y \partial_x & k_0^2\epsilon + \partial_x^2 + \partial_y \epsilon^{-1}\partial_y \epsilon
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix}
= \beta^2
\begin{pmatrix}
E_x \\
E_y
\end{pmatrix}
\]

• and it is a normal eigenvalue problem!
• analogous form for magnetic fields
Quasi TE modes

- If waveguides are broad: $\partial_y \epsilon \approx \epsilon \partial_y$
- $E_x \approx 0$
- this results in the quasi TE mode equation

$$\{\partial_x^2 + \partial_y^2 + k_0^2 \epsilon(x, y)\} E_y = \beta^2 E_y$$

- Helmholz equation

- with $\partial_y \epsilon \approx \epsilon \partial_y$ and $H_x \approx 0$
- quasi TM mode equation

$$\{\epsilon \partial_x \epsilon^{-1} \partial_x + \partial_y^2 + k_0^2 \epsilon(x, y)\} H_y = \beta^2 H_y$$

- only change is $\epsilon = \epsilon(x, y)$ and additional $\partial_y^2$
- and: quasi modes have $z$-components
Hermann von Helmholtz, German physicist, 1821 - 1894; Königsberg, Bonn, Heidelberg, Berlin
• Henceforth we speak about quasi-TE mode
• i. e. there is just one field component $E = E(x, y)$, the 'electric field' or the 'field', for short
• fields can be linearly combined, they form a linear space
• Power is
  \[ P = \frac{2\beta}{\omega \mu_0} \int \, dx \, dy \, |E(x, y)|^2 \]
• scalar product $(G, F) = \int \, dx \, dy \, G^*(x, y) \, F(x, y)$
• With this, the linear space of fields $E = E(x, y)$ with finite power transfer becomes a Hilbert space
• The Helholtz operator $H = \partial_x^2 + \partial_y^2 + k_0^2 \epsilon(x, y)$
• is self-adjoint:
  \[ (G, HF) = (HG, F) \]
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Overview

Mode equation

Helmholtz equation

Hilbert space

Coupled waveguides

Coupled modes

Random waveguide array

David Hilbert, German mathematician, 1862-1943; Königsberg, Göttingen
Self-adjoint operators $A$ have remarkable properties:
- $A\chi = a\chi$ guarantees that the eigenvalue $a$ is real.
- Denote by $\chi_1, \chi_2, \ldots$ the normalized eigenvectors.
- They form a Complete OrthoNormal Set (CONS).
- Meaning $(\chi_k, \chi_j) = \delta_{jk}$.
- And $\chi = \sum_j (\chi_j, \chi) \chi_j$ for all $\chi$.
- $HE = \Lambda E$ guarantees that $\Lambda$ is real.
- Usually, there are only a few modes $E_n$ with positive
  $\Lambda_n = \beta_n^2$.
- They cannot span the entire Hilbert space.
- We should add wave packets of evanescent and radiation
  modes.
- Consider $r = 1, 2, \ldots, N$ individual waveguides
- Such as a coupler or a waveguide array
- The entire system is again a many mode waveguide
- Its modes are supermodes
- If the single waveguides are well separated
  - the supermodes are given by $E_r$ with propagation constants $\beta_r$
- However, if $E_r$ and $E_s$ overlap, this will no longer be true
A random waveguide array
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Supermodes
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Mode expansion
If waveguide $r$ is the only one, it obeys
\[
\left( \frac{1}{k_0^2} \Delta + \epsilon_r \right) E_r = n_r^2 E_r
\]

Supermode is described by
\[
\left( \frac{1}{k_0^2} \Delta + \epsilon \right) E = n^2 E \text{ where } \epsilon(x, y) = \sum_r \epsilon_r(x, y)
\]

bold approximation:

\[
E(x, y) = \sum_r U_r E_r(x, y)
\]

With
\[
M_{sr} = (E_s, \left( \frac{1}{k_0^2} \Delta + \epsilon \right) E_r) \quad \text{and} \quad \Lambda_{sr} = (E_s, E_r)
\]

Solve generalized eigenvalue problem
\[
MU = \Lambda U
\]
Recall
\[ M_{sr} = (E_s, \left( \frac{1}{k_0^2} \Delta + \epsilon \right) E_r ) \text{ and } \Lambda_{sr} = (E_s, E_r) \]

Because of \((E_r, E_r) = |E_r|^2 = 1\), all diagonal elements of \(\Lambda\) are ones.

However, there are non-diagonal contributions (overlaps).

For a certain \(r\) one may write \(\epsilon = \epsilon_r + \bar{\epsilon}_r\)

where \(\bar{\epsilon}_r\) is the permittivity profile outside waveguide \(r\)

such that we may write
\[ M_{sr} = n_r^2 \Lambda_{sr} + (E_s, \bar{\epsilon}_r E_r) \]
Random waveguide array

- RA=rwga_descriptor()
- RA=rwga_single(RA)
- RA=rwga_overlap(RA)
- RA=rwga_dices(RA)
- RA=rwga_super(RA)
- RA=rwga_intensity(RA,MN)
- Anderson localization
Ground mode of a $30 \times 30$ random waveguide array. Probability for small core is 0.1.
Finite Difference Method for a super mode.