Propagation Equation

Solving the propagation equation

Modal approach

Fresnel approximation

Crank-Nicholson scheme

Transparent boundary conditions

Coupled modes

Evaluation

Mode Propagation

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Harmonic solutions

- Assume all fields to be of the form

  $F(t,x,y,z) = F(x,y,z)\,\mathrm{e}^{-\mathrm{i}\omega t}$
- Maxwell's equation say

  $\mathbf{curl}\,\mathbf{curl}\,\boldsymbol{E} + k_0^2\epsilon(x,y)\boldsymbol{E} = 0$
- For a quasi-TE mode this reduces to

  $\left\{\partial_x^2 + \partial_y^2 + \partial_z^2 + k_0^2\epsilon(x,y)\right\} E(x,y,z) = 0$
- may be further simplified for plane waves, or modes:

  $E(x,y,z) = E(x,y)\,\mathrm{e}^{\mathrm{i}\beta z}$
- resulting in the mode equation

  $\left\{\partial_x^2 + \partial_y^2 + k_0^2\epsilon(x,y)\right\} E(x,y) = \beta^2 E(x,y)$
- an eigenvalue problem for the propagation constant β

Modal approach to propagation

- Discern between cross section parameters x, y and propagation parameter z
- $E(x,y,z) = E(x,y;z)$ is a field living on the cross section x, y, it depends on the propagation distance z.
- We have to solve
$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}E(z) = \left\{\partial_x^2 + \partial_y^2 + k_0^2\epsilon(x,y)\right\}E(z) = HE(z)$$
- where H denotes the familiar Helmholtz operator, acting on the cross section coordinates x and y
- With $HE_j = \beta_j^2 E_j$ and
$$E(x,y;0) = \sum_j U_j E_j(x,y)$$
- we may write
$$E(x,y;z) = \sum_j U_j E_j(x,y)\,\mathrm{e}^{\,\mathrm{i}\beta_j z}$$
- which solves the propagation equation and the initial condition.

Pros and Cons

- *Pros*:
- the guided modes E_j and the corresponding propagation constants β_j have to be calculated.
- On a PC, this is usually done well below a minute
- the following propagation procedure is a matter of seconds
- *Cons*:
- The method does not allow for smooth changes of the permittivity $\epsilon(x, y; z)$ along the propagation direction z
- such as tapers (profile reformers) or bends
- Only guided modes are taken into account. The method does not allow for radiation losses.

Fresnel approximation

- Because propagation is described by
  $E''(z) = \left\{ \partial_x^2 + \partial_y^2 + k_0^2 \epsilon(x,y) \right\} E(z) = HE(z)$
- providing just the initial field $E(0)$ is not sufficient
- simplify the propagation equation to first order
- choose an average refractive index n
- such that for
  $E(z) = A(z)\,\mathrm{e}^{\,\mathrm{i} n k_0 z}$
- the field A(z) changes slowly with z
- i. e. $\|A''(z)\| \ll k_0 \|A'(z)\|$

Fresnel approximation (ctd.)

- Solve

$$\mathrm{e}^{-\mathrm{i}nk_0z}\left\{\partial_x^2+\partial_y^2+\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right\}\mathrm{e}^{\mathrm{i}nk_0z}A(z)$$
$$+k_0^2\epsilon(x,y)A(z)=0$$

- Neglecting A'' gives
- the Fresnel-Equation

$$-\mathrm{i}A'(z)=\frac{\partial_x^2+\partial_y^2+k_0^2\delta\epsilon}{2nk_0}A(z)$$

- where $\delta\epsilon=\delta\epsilon(x,y)=\epsilon(x,y)-n^2$
- Parabolic approximation
- Paraxial approximation
- If nk_0 is the propagation constant of a mode
- A does not vary with z

Augustain-Jean Fresnel, 1788 - 1827, French engineer and physicist, member of the French and British Academy of Science

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Explicit and implicit forward

- Fresnel equation

$$-\mathrm{i}A' = PA \text{ where } P = \frac{\partial_x^2 + \partial_y^2 + k_0^2\delta\epsilon(x,y)}{2nk_0}$$

- is formally solved by

$$A(z) = \mathrm{e}^{\mathrm{i}zP}$$

- For a propagation step h

$$A(z+h) = \mathrm{e}^{\mathrm{i}hP}A(z)$$

- Crude approximation

$$A(z+h) \approx (I + \mathrm{i}hP)A(z)$$

- Likewise

$$A(z) = \mathrm{e}^{-\mathrm{i}hP}A(z+h)$$

- approximated by

$$A(z+h) = (I - \mathrm{i}hP)^{-1}A(z)$$

Crank-Nicholson scheme

- Explicit forward is unstable, i. e. halfing h and doubling the number of steps again and again will not converge
- Implicit forward is always stable. However, each propagation step requires a linear set of equations to be solved.
- The accuracy for a finite propagation distance is $\propto h$
- Combining both methods leads to a stable scheme with accuracy $\propto h^2$

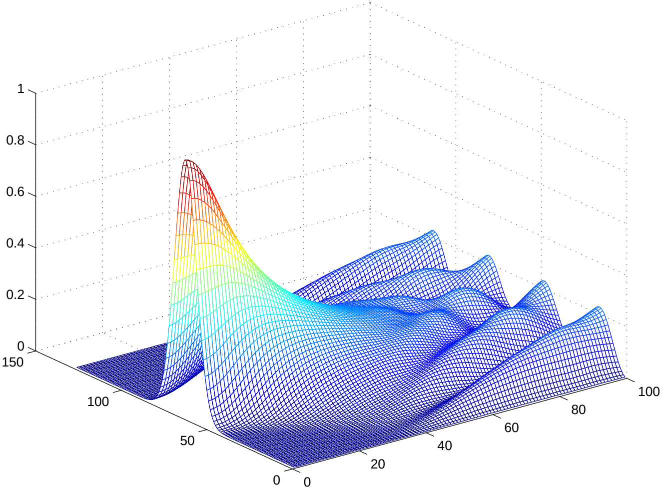
$$A(z+h/2) = (I + \mathrm{i}\frac{hP}{2})A(z) = (I - \mathrm{i}\frac{hP}{2}2)^{-1}A(z+h)$$

- that is

$$A(z+h) = \frac{1+\mathrm{i}hP/2}{1-\mathrm{i}hP/2}A(z)$$

- the Crank-Nicholson scheme of beam propagation

```
% propagation of a gaussian beam in vacuum
CW=10.0; % computational window
BW=1.0; % Gaussian beam width
LAMBDA=0.633; % helium neon laser
n=1.0; % refractive index of the medium
k0=2*pi/LAMBDA;
NX=128; % points on x axis
HX=CW/(NX-1); % x axis spacing
HZ=5*HX; % propagation step
x=linspace(-0.5*CW,0.5*CW,NX)';
A=exp(-(x/BW).^2); % initial field
u=0.5i*HZ/(2*n*k0);
main=-2*u*ones(NX,1)/HX^2;
next=u*ones(NX-1,1)/HX^2;
FW=eye(NX)+diag(next,-1)+diag(main,0)+diag(next,1);
BW=eye(NX)-diag(next,-1)-diag(main,0)-diag(next,1);
NZ=100; % number of propagation steps
hist=zeros(NX,NZ); % storage for history
for r=1:NZ
  hist(:,r)=abs(A).^2;
  A=BW\FW*A;
end;
```

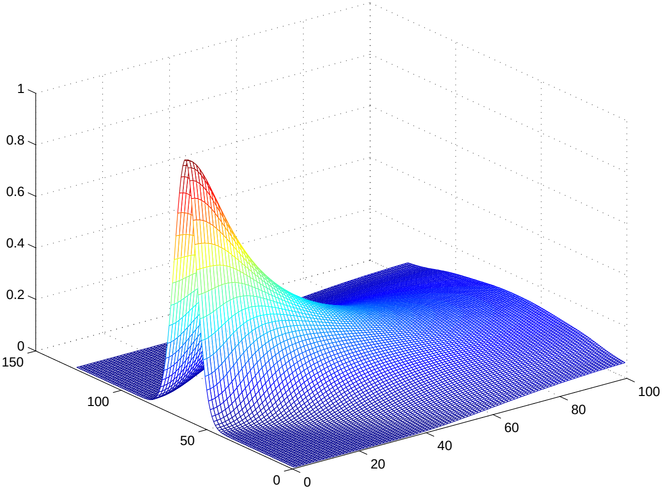


Propagation of a Gaussian beam in free space. If it hits the boundary of the computational window, it will be reflected.

Transparent boundary conditions

- The line `next=u*ones(NX-1,1)/HX^2;` assumes that the field outside the computational window vanishes
- Determine from the field values close to the boundary the amplitudes of an outgoing and an incoming wave and suppress the latter
- A^r_j represent the field $A(rh_z)$ where $j = 1, 2, \ldots, N_x$.
- Write
  $$\mathrm{e}^{\,\mathrm{i}kh_x} = \frac{A^r_N}{A^r_{N-1}}$$
- If the real part of k is positive: fine, an outgoing wave
- If the real part of k is negative, we reset it to zero and prohibit an incoming wave
- Analogous procedure at the left boundary of the computational window
- See an implementation of such transparent boundary conditions

```
function new=one_step(HX,u,FW,BW,TINY,old)
NX=size(old,1);
FF=FW; BB=BW;
if abs(old(1))>TINY
  k=i/HX*log(old(2)/old(1));
  if real(k)<0
    k=imag(k);
  end;
  tbc=exp(i*k*HX)*u/HX^2;
  FF(1,1)=FF(1,1)+tbc;
  BB(1,1)=BB(1,1)-tbc;
end;
if abs(old(NX))>TINY
  k=-i/HX*log(old(NX)/old(NX-1));
  if real(k)<0
    k=imag(k);
  end;
  tbc=exp(i*k*HX)*u/HX^2;
  FF(NX,NX)=FF(NX,NX)+tbc;
  BB(NX,NX)=BB(NX,NX)-tbc;
end;
new=BB\FF*old;
```



Propagation of a Gaussian beam in free space. Transparent boundary conditions have been implemented.

Propagation along a slab waveguide

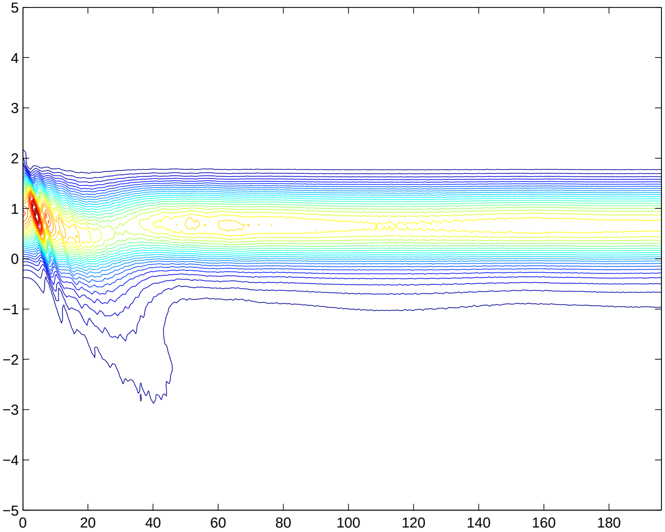
- The programs for propagation in free space have to be modified just slightly
- Recall the Fresnel equation

$$-\mathrm{i}A'(z) = \frac{\partial_x^2 + \partial_y^2 + k_0^2\delta\epsilon}{2nk_0}$$

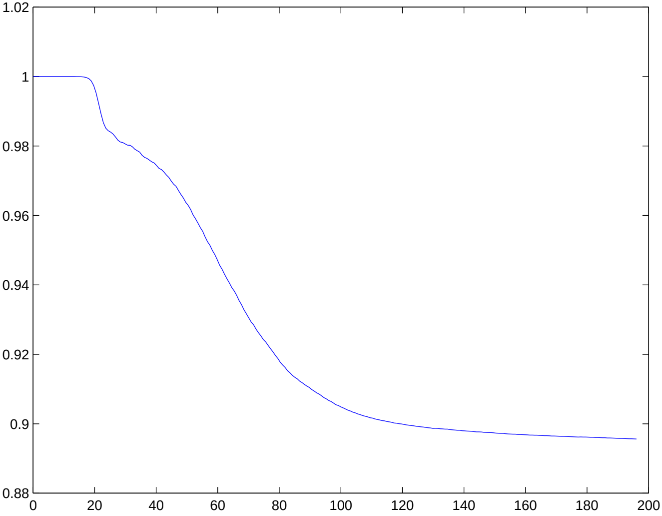
- For a planar waveguide

$$-\mathrm{i}A'(z) = \frac{\partial_x^2 + k_0^2\delta\epsilon}{2nk_0}$$

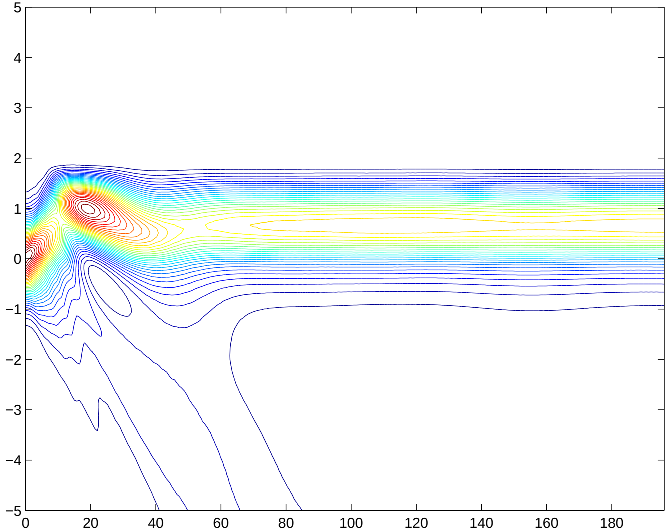
- So far $n = 1$ and $\delta\epsilon(x) = 0$ – free space
- Chose n as the substrate refractive index
- $\delta\epsilon(x)$ only modifies the main diagonal of propagation matrices
- Inspect a typical example



A Gaussian beam hits the center of the waveguide. We have plotted intensities.



Power remaining in the waveguide. The rest has been radiated off.



The center of the Gaussian beam is at the interface between substrate and film. The insertion loss is obviously larger.

Summary

- The Crank-Nicholson scheme relies on Fresnel's approximation
- It outbeats the explicit forward scheme (being unstable) and the implicit forward scheme (being stable, but converging in order h only)
- The Crank-Nicholson scheme is stable one order more accurate than its competitors
- It must be supplemented by a prescription of what happens at the boundaries of the computational window
- Hadley's prescription to suppress incoming waves solves this convincingly
- Radiation losses can be calculated
- as well as the onset of mode formation

Coupled Mode Theory

- Applicable for a structure of $r = 1, 2, \dots$ waveguides
- Couplers, waveguide arrays, one and two-dimensional, random array waveguides
- ϵ_r and β_r are permittivity and propagation constant of individual waveguide r
- i. e. solutions
  $$\{\partial_x^2 + \partial_y^2 + k_0^2 \epsilon_r(x,y)\} E_r(x,y) = \beta_r^2 E_r(x,y)$$
- Structure profile
  $$\epsilon(x,y) = \sum_r \epsilon_r(x,y)$$
- Profile outsite waveguide r is
  $$\bar{\epsilon}_r(x,y) = \epsilon(x,y) - \epsilon_r(x,y)$$

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Coupled Mode Theory (ctd.)

- For simplicity of notation: assume one mode per waveguide only
- Approximate

$$(E_r, E_s) = \int \mathrm{d}x\mathrm{d}y\, E_r^*(x,y)\, E_s(x,y) \approx \delta_{rs}$$

- Expand

$$A(x,y;z) = \sum_s U_s(z) E_s(x,y)$$

- Insert into Fresnel equation and work out the scalar product with E_s:

$$-2\mathrm{i}nk_0 U_r'(z) = (\beta_r^2 - k_0^2 n^2) U_r(z) + k_0^2 \sum_s C_{rs} U_s(z)$$

- where the matrix of coupling coefficients is

$$C_{rs} = (E_r, \bar{\epsilon}_s E_s) = \int \mathrm{d}x\mathrm{d}y\, E_r^*(x,y)\, \bar{\epsilon}_s(x,y) E_s(x,y)$$

- System of only few coupled ordinary differential equation

Initial values

- Denote by $E = E(x, y)$ the incident field at $z = 0$
- It will be represented by
$$E(x, y) = \sum_s U_s(0) E_s(x, y)$$
- Therefore
$$U_r(0) = (E_r, E) = \int \mathrm{d}x\mathrm{d}y \, E_r^*(x, y) \, E(x, y)$$
- Recall that $E_r = E_r(x, y)$ is the only guided mode of waveguide r with propagation constant β_r
- Recall the coupled mode equation
$$-2\mathrm{i}nk_0 U_r'(z) = (\beta_r^2 - k_0^2 n^2) U_r(z) + k_0^2 \sum_s C_{rs} U_s(z)$$
- n is a reference refractive which allows to justify the Fresnel approximation

Coupled modes: evaluation

- *Cons*:
- Coupled mode theory inherits the deficiencies of the quasi-TE and the quasi-TM approximation
- Coupled mode theory for propagation inherits the possible deficiencies of the Fresnel approximation
- By construction, there is no radiation lass
- *Pros*:
- From a computational point of view, coupled mode theory provides a system of only few coupled ordinary differential equations with well defined initial conditions.
- Therefore, if the structure has regularities, an analytic solution may be possible.
- Coupled mode results are often much better than expected