Space and Time

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Hilbert space

- A physical system is described by linear operators
- defined on a Hilbert space $\mathcal{H}$
- if vectors $f, g \in \mathcal{H}$, scalars $\alpha, \beta \in \mathbb{C}$, then $\alpha f + \beta g$ is a vector in $\mathcal{H}$ (linear space)
- for $f, g \in \mathcal{H}$, a scalar product (g, f) is defined
- $(h, \alpha f + \beta g) = \alpha(h, f) + \beta(h, g)$
- $(g, f) = (f, g)^*$ where * denotes complex conjugation
- $(f, f) \geq 0$ and $(f, f) = 0$ implies $f = 0$
- if $(g, f) = 0$ than g and f are orthogonal
- $\|f\| = \sqrt{(f, f)}$ defines a norm
- $\|g - f\|$ is the distance between g and f
- $\mathcal{H}$ is complete: Cauchy sequences have a limit

Operators

- linear mapping $A : \mathcal{H} \to \mathcal{H}$
- $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$
- we write $A(f) = Af$ (operator notation)
- and say *operator* if we mean a *linear operator*
- A has an inverse operator A^{-1} if $Af = 0$ is possible for $f = 0$ only
- to each operator, there is an adjoint operator $A^\dagger$
- characterized by $(g, Af) = (A^\dagger g, f)$ for all $f, g \in \mathcal{H}$
- $A^\dagger(\alpha f + \beta g) = \alpha^* A^\dagger + \beta^* B^\dagger$
- $(AB)^\dagger = B^\dagger A^\dagger$
- $\|A\|$ defined as smalles number c for which $\|Af\| \le c\|f\|$
- with operator Norm, convergence of an operator sequence $A_1, A_2, \ldots$ can be studied

Linear subspaces

- projector Π defined by $\Pi = \Pi^\dagger$ and $\Pi^2 = \Pi$
- $\mathcal{L} = \Pi\mathcal{H} = \{g \mid g = \Pi f\ ,\ f \in \mathcal{H}\}$ is linear subspace
- $g', g'' \in \mathcal{L}$, $\alpha, \beta \in \mathbb{C}$ then $g' = \Pi f', g'' = \Pi f''$
- $\alpha g' + \beta g'' = \alpha \Pi f' + \beta \Pi f'' = \Pi(\alpha f' + \beta f'')$
- projection because $\Pi\mathcal{L} = \Pi\Pi\mathcal{H} = \Pi\mathcal{H} = \mathcal{L}$
- With Π, $I - \Pi$ is also a projector
- both projectors are orthogonal: $\Pi(I - \Pi) = 0$
- both subspaces are orthogonal, $(\Pi\mathcal{H}, (I - \Pi)\mathcal{H}) = 0$
- meaning $g \in \Pi\mathcal{H}$, $f \in (I - \Pi)\mathcal{H}$, then $(g, f) = 0$
- generalization: decomposition of unity into mutually orthogonal projectors
- $I = \Pi_1 + \Pi_2 + \dots$ with
- $\Pi_i = {\Pi_i}^\dagger$ and $\Pi_i \Pi_k = \delta_{ik} \Pi_i$

Diagonalization

- choose a decomposition $I = \Pi_1 + \Pi_2 + \dots$ of unity into mutually orthogonal projectors
- construct, with complex numbers ν_i, the operator $N = \sum_i \nu_i \, \Pi_i$
- it is normal: $N N^\dagger = N^\dagger N$
- the converse is also true: every operator N which commutes with its adjoint $N^\dagger$ can written as above
- for g in $\mathcal{L}_i = \Pi_i \mathcal{H}$ we find $Ng = \nu_i g$
- the ν_i are eigenvalues, the $\mathcal{L}_i$ eigenspaces of N
- $N = \sum \nu_i \Pi_i$ is the diagonalization of the (normal) operator N
- for normal operators N, the function $f(N)$ is defined by $f(N) = \sum_i f(\nu_i) \Pi_i$

Self-adjoint and unitary operators

- self-adjoint operators are characterized by $M = M^\dagger$
- self-adjoint operators are normal
- their eigenvalues are real numbers
- unitary operators are defined by $U^\dagger U = I$
- $\|f\|^2 = (f, f) = (f, U^\dagger U f) = (Uf, Uf) = \|Uf\|^2$
- only the zero vector is mapped into the zero vector
- U has an inverse U^{-1}, $U = U^\dagger$, $UU^\dagger = I$
- U is normal and can be diagonalized
- its eigenvalues lie on the unit circle in the complex plain

Automorphisms

- unitary transformation of operators by $\bar{A} = UAU^\dagger$
- $\overline{\alpha A} = \alpha \bar{A}$ and $\overline{A+B} = \bar{A} + \bar{B}$
- $\overline{BA} = \bar{B}\bar{A}$ and $\overline{A^\dagger} = \bar{A}^\dagger$
- $\bar{A} = UAU^\dagger$ preserves the structure of the operator algebra
- note: 'normal' remains normal
- note: eigenvalues are not changed
- note: commutator $[A, B] = \mathrm{i}C$ becomes $[\bar{A}, \bar{B}] = \mathrm{i}\bar{C}$
- If $U = U(x)$ and $U(x+y) = U(x)U(y)$ then $U(x) = \mathrm{e}^{\mathrm{i}xG}$ with $G = G^\dagger$
- the $U(x)$ form an Abelian one-parameter group which is generated by the self-adjoint generator G
- conversely: every structure preserving mapping $A \to \bar{A}$ is described by $\bar{A} = UAU^\dagger$ with unitary U

Energy

- measure M after the system has been prepared
- wait for time span t and measure M is described by M_t
- automorphism $M_t = U_t M U^\dagger{}_t$
- $U_{t''+t'} = U_{t''} U_{t'}$
- this one-dimensional Abelian group is described by
  $U_t = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} tH}$
- the generator H is the system's energy
- energy is an observable
- note $U^\dagger{}_t = U_{-t}$
- $M_t = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} tH} M \, \mathrm{e}^{+\frac{\mathrm{i}}{\hbar} tH}$
- Heisenberg equation of motion
  $\frac{\mathrm{d}M_t}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar} [H, M_t]$
- $[H, H] = 0$ says: energy is conserved

Linear momentum

- translate the entire system by $\boldsymbol{a}$
- suppose that external fields are translated as well
- by the same arguments as above
  $M_{\boldsymbol{a}} = U_{\boldsymbol{a}} M U_{-\boldsymbol{a}}$
- where
  $U_{\boldsymbol{a}} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\boldsymbol{a}\cdot\boldsymbol{P}}$
- $\boldsymbol{P}$ is the linear momentum, three observables
- since translations commute:
  $[P_i, P_k] = 0$
- time and space translations commute, therefore
  $[H, P_i] = 0$
- linear momentum is conserved

Location

- a location observable $\boldsymbol{X}$ is characerized by
  $U_{\boldsymbol{a}}\boldsymbol{X}U_{-\boldsymbol{a}} = \boldsymbol{X} + \boldsymbol{a}$
- small translation
  $(I + \frac{\mathrm{i}}{\hbar}\, a_j P_j + \dots) X_k (I - \frac{\mathrm{i}}{\hbar}\, a_j P_j + \dots) = a_k$
- canonical commutation relations
  $[P_j, X_k] = \frac{\hbar}{\mathrm{i}}\, \delta_{kj}\, I$
- uncertainty principle: momentum and location cannot be measured simultaneously
- however, different components of momentum can be measured simultaneously
- plausible: different components of location as well
- therefore
  $[X_j, X_k] = 0$

Angular momentum

- rotate the system by α_1 around the 1-axis
- generated by angular momentum J_1

  $U_{\alpha_1} = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\alpha_1 J_1}$
- the same for rotations about the 2, and 3 axes
- angular momentum components do not commute, instead
- $[J_i, J_j] = \mathrm{i}\hbar\,\epsilon_{ijk}\,J_k$
- orbital angular momentum $\boldsymbol{L} = \boldsymbol{X} \times \boldsymbol{P}$
- $[L_i, L_j] = i\hbar L_k$
- Spin (or internal angular momentum) is defined by $\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S}$
- $\boldsymbol{P}$, $\boldsymbol{X}$, $\boldsymbol{J}$, $\boldsymbol{L}$ ans $\boldsymbol{S}$ transform as vectors $\boldsymbol{V}$
- $[J_i, V_j] = \mathrm{i}\hbar\epsilon_{ijk}V_k$
- scalars S behave as $[J_i, S] = 0$

Summary

$[A,B]$	H	P_j	X_j	J_j	L_j	S_j
H	0	0	$*$	0	$*$	$*$
P_i	0	0	$-\mathrm{i}\hbar\delta_{ij}I$	$\mathrm{i}\hbar\epsilon_{ijk}P_k$	$\mathrm{i}\hbar\epsilon_{ijk}P_k$	0
X_i	$*$	$\mathrm{i}\hbar\delta_{ij}I$	0	$\mathrm{i}\hbar\epsilon_{ijk}X_k$	$\mathrm{i}\hbar\epsilon_{ijk}X_k$	0
J_i	0	$\mathrm{i}\hbar\epsilon_{ijk}P_k$	$\mathrm{i}\hbar\epsilon_{ijk}X_k$	$\mathrm{i}\hbar\epsilon_{ijk}J_k$	$\mathrm{i}\hbar\epsilon_{ijk}L_k$	$\mathrm{i}\hbar\epsilon_{ijk}S_k$
L_i	$*$	$\mathrm{i}\hbar\epsilon_{ijk}P_k$	$\mathrm{i}\hbar\epsilon_{ijk}X_k$	$\mathrm{i}\hbar\epsilon_{ijk}L_k$	$\mathrm{i}\hbar\epsilon_{ijk}L_k$	0
S_i	$*$	0	0	$\mathrm{i}\hbar\epsilon_{ijk}S_k$	0	$\mathrm{i}\hbar\epsilon_{ijk}S_k$

- commutators $[A,B]$ for A running in rows and B in columns
- asterisk: entry depends on Hamiltonian
- zeros in H column/row indicate constants of motion
- $\mathbb{R}^3$ has only two constant tensors, namely δ_{ij} and ϵ_{ijk}. No wonder we meet them here.